MAJOR ARCS FOR GOLDBACH'S PROBLEM

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Abstract. The ternary Goldbach conjecture, or three-primes problem, asserts that every odd integer n greater than 5 is the sum of three primes. The present paper proves this conjecture.

Both the ternary Goldbach conjecture and the binary, or strong, Goldbach conjecture had their origin in an exchange of letters between Euler and Goldbach in 1742. We will follow an approach based on the circle method, the large sieve and exponential sums, supplemented by rigorous computations, including a verification of zeros of L-functions due to D. Platt. The improved estimates on exponential sums are proven in a twin paper by the author.

Contents

1. Introduction

1.1. **Results.** The ternary Goldbach conjecture (or *three-prime problem*) states that every odd number n greater than 5 can be written as the sum of three primes. Both the ternary Goldbach conjecture and the (stronger) binary Goldbach conjecture (stating that every even number greater than 2 can be written as the sum of two primes) have their origin in the correspondence between Euler and Goldbach (1742).

I. M. Vinogradov [Vin37] showed in 1937 that the ternary Goldbach conjecture is true for all n above a large constant C. Unfortunately, while the value of C has been improved several times since then, it has always remained much too large ($C = e^{3100}$, [LW02]) for a mechanical verification up to C to be even remotely feasible.

The present paper proves the ternary Goldbach conjecture.

Main Theorem. *Every odd integer n greater than 5 can be expressed as the sum of three primes.*

The proof given here works for all $n \geq C = 10^{30}$. The main theorem has been checked deterministically by computer for all $n < 10^{30}$ (and indeed for all $n \leq 8.875 \cdot 10^{30}$) [HP].

We are able to set major arcs to be few and narrow because the minor-arc estimates in [Hel] are very strong; we are forced to take them to be few and narrow because of the kind of L-function bounds we will rely upon.

At issue are

(1) a fuller use of the close relation between the circle method and the large sieve;
(2) a combination of different smoothings for different tasks;
(3) the verification of GRH up to a bounded height for all conductors $q \leq 150000$ and all even conductors $q \leq 300000$ (due to David Platt [Plab]);
(4) better bounds for exponential sums, as in [Hel].

All major computations – including D. Platt's work in [Plab] – have been conducted rigorously, using interval arithmetic.

1.2. **History.** The following brief remarks are here to provide some background; no claim to completeness is made. Results on exponential sums over the primes are discussed more specifically in [Hel, §1].

1.2.1. *Results towards the ternary Goldbach conjecture.* Hardy and Littlewood [HL23] proved that every odd number larger than a constant C is the sum of three primes, conditionally on the generalized Riemann Hypothesis. This showed, as they said, that the problem was not *unergreifbar* (as it had been called by Landau in [Lan12]).

Vinogradov [Vin37] made the result unconditional. An explicit value for C (namely, $C = 3^{3^{15}}$) was first found by Borodzin in 1939. This value was improved to $C = 3.33 \cdot 10^{43000}$ by J.-R. Chen and T. Z. Wang [CW89] and to $C = 2 \cdot 10^{1346}$ by M.-Ch. Liu and T. Wang [LW02]. (J.-R. Chen had also proven that every large enough even number is either the sum of two primes or the sum $p_1 + p_2 p_3$ of a prime p_1 and the product $p_2 p_3$ of two primes.)

In [DEtRZ97], the ternary Goldbach conjecture was proven for all n conditionally on the generalized Riemann hypothesis.

1.2.2. *Checking Goldbach for small n.* Numerical verifications of the binary Goldbach conjecture for small n were published already in the late nineteenth century; see [Dic66, Ch. XVIII]. Richstein [Ric01] showed that every even integer $4 \leq n \leq 4 \cdot 10^{14}$ is the sum of two primes. Oliveira e Silva, Herzog and Pardi [OeSHP13] have proven that every even integer $4 \leq n \leq 4 \cdot 10^{18}$ is the sum of two primes.

The question is then until what point one can establish the ternary Goldbach conjecture using [OeSHP13]. Clearly, if one can show that every interval of length $\geq 4 \cdot 10^{18}$ within $[1, N]$ contains a prime, then [OeSHP13] implies that every odd number between 7 and N can be written as the sum of three primes. This was used in a first version of [Hel] to show that the best existing result on prime gaps ([RS03], with [Plaa] as input) implies that every odd number between 7 and $1.23 \cdot 10^{27}$ is the sum of three primes. A more explicit approach to prime gaps [HP] now shows that every odd integer $7 \leq n \leq 8.875694 \cdot 10^{30}$ is the sum of three primes.

1.2.3. *Work on Schnirelman's constant.* "Schnirelman's constant" is a term for the smallest k such that every integer $n > 1$ is the sum of at most k primes. (Thus, Goldbach's binary and ternary conjecture, taken together, are equivalent to the statement that Schnirelman's constant is 3.) In 1930, Schnirelman [Sch33] showed that Schnirelman's constant k is finite, developing in the process some of the bases of what is now called additive or arithmetic combinatorics.

In 1969, Klimov proved that $k \leq 6 \cdot 10^9$; he later improved this result to $k \leq 115$ [KPŠ72] (with G. Z. Piltay and T. A. Sheptiskaya) and $k \leq 55$. Results by Vaughan [Vau77] ($k = 27$), Deshouillers [Des77] ($k = 26$) and Riesel-Vaughan [RV83] ($k = 19$) then followed.

Ramaré showed in 1995 that every even $n > 1$ is the sum of at most 6 primes [Ram95]. Recently, Tao [Tao] established that every odd number $n > 1$ is the sum of at most 5 primes. These results imply that $k \leq 6$ and $k \leq 5$, respectively. The present paper implies that $k \leq 4$.

1.2.4. *Other approaches.* Since [HL23] and [Vin37], the main line of attack on the problem has gone through exponential sums. There are proofs based on cancellation in other kinds of sums ([HB85], [IK04, §19]), but they have not been made to yield practical estimates. The same goes for proofs based on other principles, such as that of Schnirelman's result or the recent work of X. Shao

[Sha]. (It deserves to be underlined that [Sha] establishes Vinogradov's three-prime result without using L-function estimates at all; the constant C is, however, extremely large.)

1.3. **Main ideas.** We will limit the discussion here to the general setup, the estimates for major arcs and the efficient usage of exponential-sum estimates on the minor arcs. The development of new exponential-sum estimates is the subject of [Hel].

In the circle method, the number of representations of a number N as the sum of three primes is represented as an integral over the "circle" $\mathbb{R}/\mathbb{Z}$, which is partitioned into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m} = (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}$:

$$\begin{aligned}\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) &= \int_{\mathbb{R}/\mathbb{Z}} (S(\alpha,x))^3 e(-N\alpha)d\alpha \\ &= \int_{\mathfrak{M}} (S(\alpha,x))^3 e(-N\alpha)d\alpha + \int_{\mathfrak{m}} (S(\alpha,x))^3 e(-N\alpha)d\alpha,\end{aligned} \tag{1.1}$$

where $S(\alpha,x) = \sum_{n=1}^{x} \Lambda(n)e(\alpha n)$. The aim is to show that the sum of the integral over $\mathfrak{M}$ and the integral over $\mathfrak{m}$ is positive; this will prove the three-primes theorem.

The major arcs $\mathfrak{M} = \mathfrak{M}_{r_0}$ consist of intervals $(a/q - cr_0/qx, a/q + cr_0/qx)$ around the rationals a/q, $q \leq r_0$, where c is a constant. In previous work[1], r_0 grew with x; in our setup, r_0 is a constant. Smoothing changes the left side of (1.1) into a weighted sum, but, since we aim at an existence result rather than at an asymptotic for the number of representations $p_1 + p_2 + p_3$ of N, this is obviously acceptable.

Typically, work on major arcs yields rather precise estimates on the integral over $\int_{\mathfrak{M}}$ in (1.1), whereas work on minor arcs gives upper bounds on the absolute value of the integral over $\int_{\mathfrak{m}}$ in (1.1). Exponential-sum estimates, such as those in [Hel], provide upper bounds for $\max_{\alpha\in\mathfrak{m}} |S(\alpha,x)|$.

1.3.1. *Major arc bounds.* We will be working with smoothed sums

$$S_\eta(\alpha,x) = \sum_{n=1}^{\infty} \Lambda(n)\chi(n)e(\delta n/x)\eta(n/x). \tag{1.2}$$

Our integral will actually be of the form

$$\int_{\mathfrak{M}} S_{\eta_+}(\alpha,x)^2 S_{\eta_*}(\alpha,x)e(-N\alpha)d\alpha, \tag{1.3}$$

where η_+ and η_* are two different smoothing functions to be discussed soon.

Estimating the sums (1.2) on $\mathfrak{M}$ reduces to estimating the sums

$$S_\eta(\delta/x,x) = \sum_{n=1}^{\infty} \Lambda(n)\chi(n)e(\delta n/x)\eta(n/x) \tag{1.4}$$

for χ varying among all Dirichlet characters modulo $q \leq r_0$ and for $|\delta| \leq cr_0/q$, i.e., $|\delta|$ small. Using estimates on (1.4) efficiently in the estimation of (1.3) is a delicate task; this is the subject of §3. Let us now focus on how to obtain estimates on (1.4).

[1]Ramaré's work [Ram10] is in principle strong enough to allow r_0 to be an unspecified large constant. Tao's work [Tao] reaches this standard only for x of moderate size.

Sums such as (1.4) are estimated using Dirichlet L-functions $L(s,\chi)$. An *explicit formula* gives an expression

$$S_{\eta,\chi}(\delta/x,x) = I_{q=1}\widehat{\eta}(-\delta)x - \sum_\rho F_\delta(\rho)x^\rho + \text{small error}, \tag{1.5}$$

where $I_{q=1} = 1$ if $q = 1$ and $I_{q=1} = 0$ otherwise. Here ρ runs over the complex numbers ρ with $L(\rho,\chi) = 0$ and $0 < \Re(\rho) < 1$ ("non-trivial zeros"). The function F_δ is the Mellin transform of $e(\delta t)\eta(t)$ (see §2.4).

The questions are then: where are the non-trivial zeros ρ of $L(s,\chi)$? How fast does $F_\delta(\rho)$ decay as $\Im(\rho) \to \pm\infty$?

Write $\sigma = \Re(s)$, $\tau = \Im(s)$. The belief is, of course, that $\sigma = 1/2$ for every non-trivial zero (Generalized Riemann Hypothesis), but this is far from proven. Most work to date has used zero-free regions of the form $\sigma \leq 1 - 1/C\log q|\tau|$, C a constant. This is a classical zero-free region, going back, qualitatively, to de la Vallée-Poussin (1899). The best values of C known are due to McCurley [McC84] and Kadiri [Kad05].

These regions seem too narrow to yield a proof of the three-primes theorem. What we will use instead is a finite verification of GRH "up to T_q", i.e., a computation showing that, for every Dirichlet character of conductor $q \leq r_0$ (r_0 a constant, as above), every non-trivial zero $\rho = \sigma + i\tau$ with $|\tau| \leq T_q$ satisfies $\Re(\sigma) = 1/2$. Such verifications go back to Riemann; modern computer-based methods are descended in part from a paper by Turing [Tur53]. (See the historical article [Boo06].) In his thesis [Pla11], D. Platt gave a rigorous verification for $r_0 = 10^5$, $T_q = 10^8/q$. In coordination with the present work, he has extended this to

- all odd $q \leq 3\cdot 10^5$, with $T_q = 10^8/q$,
- all even $q \leq 4\cdot 10^5$, with $T_q = \max(10^8/q, 200 + 7.5\cdot 10^7/q)$.

This was a major computational effort, involving, in particular, a fast implementation of interval arithmetic (used for the sake of rigor).

What remains to discuss, then, is how to choose η in such a way $F_\delta(\rho)$ decreases fast enough as $|\tau|$ increases, so that (1.5) gives a good estimate. We cannot hope for $F_\delta(\rho)$ to start decreasing consistently before $|\tau|$ is at least as large as a multiple of $2\pi|\delta|$. Since δ varies within $(-cr_0/q, cr_0/q)$, this explains why T_q is taken inversely proportional to q in the above. As we will work with $r_0 \geq 150000$, we also see that we have little margin for maneuver: we want $F_\delta(\rho)$ to be extremely small already for, say, $|\tau| \geq 80|\delta|$. We also have a Scylla-and-Charybdis situation, courtesy of the uncertainty principle: roughly speaking, $F_\delta(\rho)$ cannot decrease faster than exponentially on $|\tau|/|\delta|$ both for $|\delta| \leq 1$ and for δ large.

The most delicate case is that of δ large, since then $|\tau|/|\delta|$ is small. It turns out we can manage to get decay that is much faster than exponential for δ large, while no slower than exponential for δ small. This we will achieve by working with smoothing functions based on the (one-sided) Gaussian $\eta_\heartsuit(t) = e^{-t^2/2}$.

The Mellin transform of the twisted Gaussian $e(\delta t)e^{-t^2/2}$ is a parabolic cylinder function $U(a,z)$ with z purely imaginary. Since fully explicit estimates for $U(a,z)$, z imaginary, have not been worked in the literature, we will have to derive them ourselves. This is the subject of §5.

We still have some freedom in choosing η_+ and η_*, though they will have to be based on $\eta_\heartsuit(t) = e^{-t^2/2}$. The main term in our estimate for (1.3) is of the form

$$C_0 \int_0^\infty \int_0^\infty \eta_+(t_1)\eta_+(t_2)\eta_*\left(\frac{N}{x} - (t_1+t_2)\right) dt_1 dt_2, \tag{1.6}$$

where C_0 is a constant. Our upper bound for the minor-arc integral, on the other hand, will be proportional to $|\eta_+|_2^2|\eta_*|_1$. The question is then how to make (1.6) divided by $|\eta_+|_2^2|\eta_*|_1$ as large as possible. A little thought will show that it is best for η_+ to be symmetric, or nearly symmetric, around $t = 1$ (say), and for η_* be concentrated on a much shorter interval than η_+, while x is set to be $x/2$ or slightly less.

It is easy to construct a function of the form $t \mapsto h(t)\eta_\heartsuit(t)$ symmetric around $t = 1$, with support on $[0, 2]$. We will define $\eta_+ = h_H(t)\eta_\heartsuit(t)$, where h_H is an approximation to h that is band-limited in the Mellin sense. This will mean that the decay properties of the Mellin transform of $e(\delta t)\eta_+(t)$ will be like those of $e(\delta t)\eta_\heartsuit(t)$, i.e., very good.

How to choose η_*? The bounds in [Hel] were derived for $\eta_2 = (2I_{[1/2,1]}) *_M (2I_{[1/2,1]})$, which is nice to deal with in the context of combinatorially flavored analytic number theory, but it has a Mellin transform that decays much too slowly.[2] The solution is to use a smoothing that is, so to speak, Janus-faced, viz., $\eta_* = (\eta_2 *_M \phi)(\varkappa t)$, where $\phi(t) = t^2 e^{-t^2/2}$ and $\varkappa$ is a large constant. We estimate sums of type $S_\eta(\alpha, x)$ by estimating $S_{\eta_2}(\alpha, x)$ if $S_{\eta_2}(\alpha, x)$ if α lies on a minor arc, or by estimating $S_\phi(\alpha, x)$ if α lies on a major arc. (The Mellin transform of ϕ is just a shift of that of $\eta_\heartsuit$.) This is possible because η_2 has support bounded away from zero, while ϕ is also concentrated away from 0.

1.3.2. *Minor arc bounds: exponential sums and the large sieve.* Let $\mathfrak{m}_r$ be the complement of $\mathfrak{M}_r$. In particular, $\mathfrak{m} = \mathfrak{m}_{r_0}$ is the complement of $\mathfrak{M} = \mathfrak{M}_{r_0}$. Exponential sum-estimates, such as those in [Hel], give bounds on $\max_{\alpha\in\mathfrak{m}_r} |S(\alpha, x)|$ that decrease with r.

We need to do better than

$$\begin{aligned}\int_{\mathfrak{m}} \left|S(\alpha,x)^3 e(-N\alpha)\right| d\alpha &\leq (\max_{\alpha\in\mathfrak{m}} |S(\alpha,x)|_\infty) \cdot \int_{\mathfrak{m}} |S(\alpha,x)|^2 d\alpha \\ &\leq (\max_{\alpha\in\mathfrak{m}} |S(\alpha,x)|_\infty) \cdot \left(|S|_2^2 - \int_{\mathfrak{M}} |S(\alpha,x)|^2 d\alpha\right),\end{aligned} \tag{1.7}$$

as this inequality involves a loss of a factor of $\log x$ (because $|S|_2^2 \sim x \log x$). Fortunately, minor arc estimates are valid not just for a fixed r_0, but for the complement of $\mathfrak{M}_r$, where r can vary within a broad range. By partial summation, these estimates can be combined with upper bounds for

$$\int_{\mathfrak{M}_r} |S(\alpha,x)|^2 d\alpha - \int_{\mathfrak{M}_{r_0}} |S(\alpha,x)|^2 d\alpha.$$

Giving an estimate for the integral over $\mathfrak{M}_{r_0}$ (r_0 a constant) will be part of our task over the major arcs. The question is how to give an upper bound for the integral over $\mathfrak{M}_r$ that is valid and non-trivial over a broad range of r.

[2] This parallels the situation in the transition from Hardy and Littlewood [HL23] to Vinogradov [Vin37]. Hardy and Littlewood used the smoothing $\eta(t) = e^{-t}$, whereas Vinogradov used the brusque (non-)smoothing $\eta(t) = I_{[0,1]}$. Arguably, this is not just a case of technological decay; $I_{[0,1]}$ has compact support and is otherwise easy to deal with in the minor-arc regime.

The answer lies in the deep relation between the circle method and the large sieve. (This was obviously not available to Vinogradov in 1937; the large sieve is a slightly later development (Linnik [Lin41], 1941) that was optimized and fully understood later still.) A large sieve is, in essence, an inequality giving a discretized version of Plancherel's identity. Large sieves for primes show that the inequality can be sharpened for sequences of prime support, provided that, on the Fourier side, the sum over frequencies is shortened. The idea here is that this kind of improvement can be adapted back to the continuous context, so as to give upper bounds on the L_2 norms of exponential sums with prime support when α is restricted to special subsets of the circle. Such an L_2 norm is nothing other than $\int_{\mathfrak{M}_r} |S(\alpha, x)|^2 d\alpha$.

The first version of [Hel] used an idea of Heath-Brown's[3] that can indeed be understood in this framework. In §7.1, we shall prove a better bound, based on a large sieve for primes due to Ramaré [Ram09]. We will re-derive this sieve using an idea of Selberg's. We will then make it fully explicit in the crucial range (7.2). (This, incidentally, also gives fully explicit estimates for Ramaré's large sieve in its original discrete context, making it the best large sieve for primes in a wide range.)

The outcome is that $\int_{\mathfrak{M}_r} |S(\alpha, x)|^2 d\alpha$ is bounded roughly by $2x \log r$, rather than by $x \log x$ (or by $2e^\gamma x \log r$, as was the case when Heath-Brown's idea was used). The lack of a factor of $\log x$ makes it possible to work with r_0 equal to a constant, as we have done; the factor of e^γ reduces the need for computations by more than an order of magnitude.

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The present work would most likely not have been possible without free and publicly available software: PARI, Maxima, Gnuplot, VNODE-LP, PROFIL / BIAS, SAGE, and, of course, LaTeX, Emacs, the gcc compiler and GNU/Linux in general. Some exploratory work was done in SAGE and Mathematica. Rigorous calculations used either D. Platt's interval-arithmetic package (based in part on Crlibm) or the PROFIL/BIAS interval arithmetic package underlying VNODE-LP.

The calculations contained in this paper used a nearly trivial amount of resources; they were all carried out on the author's desktop computers at home

[3]Communicated by Heath-Brown to the author, and by the author to Tao, as acknowledged in [Tao]. The idea is based on a lemma by Montgomery (as in, e.g., [IK04, Lemma 7.15]).

and work. However, D. Platt's computations [Plab] used a significant amount of resources, kindly donated to D. Platt and the author by several institutions. This crucial help was provided by MesoPSL (affiliated with the Observatoire de Paris and Paris Sciences et Lettres), Université de Paris VI/VII (UPMC - DSI - Pôle Calcul), University of Warwick (thanks to Bill Hart), University of Bristol, France Grilles (French National Grid Infrastructure, DIRAC instance), Université de Lyon 1 and Université de Bordeaux 1. Both D. Platt and the author would like to thank the donating organizations, their technical staff, and all academics who helped to make these resources available to us.

2. Preliminaries

2.1. **Notation.** As is usual, we write μ for the Moebius function, Λ for the von Mangoldt function. We let $\tau(n)$ be the number of divisors of an integer n and $\omega(n)$ the number of prime divisors. For p prime, n a non-zero integer, we define $v_p(n)$ to be the largest non-negative integer α such that $p^\alpha|n$.

We write (a,b) for the greatest common divisor of a and b. If there is any risk of confusion with the pair (a,b), we write $\gcd(a,b)$. Denote by (a,b^∞) the divisor $\prod_{p|b} p^{v_p(a)}$ of a. (Thus, $a/(a,b^\infty)$ is coprime to b, and is in fact the maximal divisor of a with this property.)

As is customary, we write $e(x)$ for $e^{2\pi i x}$. We write $|f|_r$ for the L_r norm of a function f. Given $x \in \mathbb{R}$, we let

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We write $O^*(R)$ to mean a quantity at most R in absolute value.

2.2. **Dirichlet characters and L functions.** A *Dirichlet character* $\chi : \mathbb{Z} \to \mathbb{C}$ of modulus q is a character χ of $(\mathbb{Z}/q\mathbb{Z})^*$ lifted to $\mathbb{Z}$ with the convention that $\chi(n) = 0$ when $(n,q) \neq 1$. Again by convention, there is a Dirichlet character of modulus $q = 1$, namely, the *trivial character* $\chi_T : \mathbb{Z} \to \mathbb{C}$ defined by $\chi_T(n) = 1$ for every $n \in \mathbb{Z}$.

If χ is a character modulo q and χ' is a character modulo $q'|q$ such that $\chi(n) = \chi'(n)$ for all n coprime to q, we say that χ' *induces* χ. A character is *primitive* if it is not induced by any character of smaller modulus. Given a character χ, we write χ^* for the (uniquely defined) primitive character inducing χ. If a character χ mod q is induced by the trivial character χ_T, we say that χ is *principal* and write χ_0 for χ (provided the modulus q is clear from the context). In other words, $\chi_0(n) = 1$ when $(n,q) = 1$ and $\chi_0(n) = 0$ when $(n,q) = 0$.

A Dirichlet L-function $L(s,\chi)$ (χ a Dirichlet character) is defined as the analytic continuation of $\sum_n \chi(n) n^{-s}$ to the entire complex plane; there is a pole at $s = 1$ if χ is principal.

A non-trivial zero of $L(s,\chi)$ is any $s \in \mathbb{C}$ such that $L(s,\chi) = 0$ and $0 < \Re(s) < 1$. (In particular, a zero at $s = 0$ is called "trivial", even though its contribution can be a little tricky to work out. The same would go for the other zeros with $\Re(s) = 0$ occuring for χ non-primitive, though we will avoid this issue by working mainly with χ primitive.) The zeros that occur at (some) negative integers are called *trivial zeros.*

The *critical line* is the line $\Re(s) = 1/2$ in the complex plane. Thus, the generalized Riemann hypothesis for Dirichlet L-functions reads: for every Dirichlet character χ, all non-trivial zeros of $L(s,\chi)$ lie on the critical line. Verifiable finite versions of the generalized Riemann hypothesis generally read: for every Dirichlet character χ of modulus $q \leq Q$, all non-trivial zeros of $L(s,\chi)$ with $|\Im(s)| \leq f(q)$ lie on the critical line (where $f : \mathbb{Z} \to \mathbb{R}^+$ is some given function).

2.3. **Fourier transforms.** The Fourier transform on $\mathbb{R}$ is normalized as follows:

$$\widehat{f}(t) = \int_{\infty}^{\infty} e(-xt) f(x) dx$$

for $f : \mathbb{R} \to \mathbb{C}$.

The trivial bound is $|\widehat{f}|_\infty \leq |f|_1$. Integration by parts gives that, if f is differentiable k times outside finitely many points, then

$$\widehat{f}(t) = O^*\left(\frac{|\widehat{f^{(k)}}|_\infty}{2\pi t}\right) = O^*\left(\frac{|f^{(k)}|_1}{(2\pi t)^k}\right). \tag{2.1}$$

It could happen that $|f^{(k)}|_1 = \infty$, in which case (2.1) is trivial (but not false). In practice, we require $f^{(k)} \in L_1$. In a typical situation, f is differentiable k times except at $x_1, x_2, \ldots, x_k$, where it is differentiable only $(k-2)$ times; the contribution of x_i (say) to $|f^{(k)}|_1$ is then $|\lim_{x\to x_i^+} f^{(k-1)}(x) - \lim_{x\to x_i^-} f^{(k-1)}(x)|$.

2.4. **Mellin transforms.** The *Mellin transform* of a function $\phi : (0,\infty) \to \mathbb{C}$ is

$$M\phi(s) := \int_0^\infty \phi(x) x^{s-1} dx. \tag{2.2}$$

In general, $M(f *_M g) = Mf \cdot Mg$ and

$$M(f\cdot g)(s) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma+i\infty} Mf(z) Mg(s-z) dz \qquad \text{[GR00, §17.32]} \tag{2.3}$$

provided that z and $s-z$ are within the strips on which Mf and Mg (respectively) are well-defined.

The Mellin transform is an isometry, in the sense that

$$\int_0^\infty |f(t)|^2 t^{2\sigma} \frac{dt}{t} = \frac{1}{2\pi}\int_{-\infty}^\infty |Mf(\sigma+it)|^2 dt. \tag{2.4}$$

provided that $\sigma + i\mathbb{R}$ is within the strip on which Mf is defined. We also know that, for general f,sear

$$\begin{aligned} M(tf'(t))(s) &= -s \cdot Mf(s), \\ M((\log t) f(t))(s) &= (Mf)'(s) \end{aligned} \tag{2.5}$$

(as in, e.g., [BBO10, Table 1.11]).

Since (see, e.g., [BBO10, Table 11.3] or [GR00, §16.43])

$$(MI_{[a,b]})(s) = \frac{b^s - a^s}{s},$$

we see that

$$M\eta_2(s) = \left(\frac{1-2^{-s}}{s}\right)^2, \qquad M\eta_4(s) = \left(\frac{1-2^{-s}}{s}\right)^4. \tag{2.6}$$

Let $f_z = e^{-zt}$, where $\Re(z) > 0$. Then

$$\begin{aligned}(Mf)(s) &= \int_0^\infty e^{-zt} t^{s-1} dt = \frac{1}{z^s}\int_0^\infty e^{-t} dt\\ &= \frac{1}{z^s}\int_0^{z\infty} e^{-u} u^{s-1} du = \frac{1}{z^s}\int_0^\infty e^{-t} t^{s-1} dt = \frac{\Gamma(s)}{z^s},\end{aligned}$$

where the next-to-last step holds by contour integration, and the last step holds by the definition of the Gamma function $\Gamma(s)$.

3. Preparatory work on major arcs

Let

$$S_\eta(\alpha, x) = \sum_n \Lambda(n) e(\alpha n) \eta(n/x), \tag{3.1}$$

where $\alpha \in \mathbb{R}/\mathbb{Z}$, Λ is the von Mangoldt function and $\eta : \mathbb{R} \to \mathbb{C}$ is of fast enough decay for the sum to converge.

Our ultimate goal is to bound from below

$$\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_1(n/x)\eta_2(n/x)\eta_3(n/x), \tag{3.2}$$

where $\eta_1, \eta_2, \eta_3 : \mathbb{R} \to \mathbb{C}$. As can be readily seen, (3.2) equals

$$\int_{\mathbb{R}/\mathbb{Z}} S_{\eta_1}(\alpha, x) S_{\eta_2}(\alpha, x) S_{\eta_3}(\alpha, x) e(-N\alpha) d\alpha. \tag{3.3}$$

In the circle method, the set $\mathbb{R}/\mathbb{Z}$ gets partitioned into the set of *major arcs* $\mathfrak{M}$ and the set of *minor arcs* $\mathfrak{m}$; the contribution of each of the two sets to the integral (3.3) is evaluated separately.

Our object here is to treat the major arcs: we wish to estimate

$$\int_{\mathfrak{M}} S_{\eta_1}(\alpha, x) S_{\eta_2}(\alpha, x) S_{\eta_3}(\alpha, x) e(-N\alpha) d\alpha \tag{3.4}$$

for $\mathfrak{M} = \mathfrak{M}_{\delta_0, r}$, where

(3.5)
$$\mathfrak{M}_{\delta_0,r} = \bigcup_{\substack{q \le r\\ q \text{ odd}}} \bigcup_{\substack{a \bmod q\\ (a,q)=1}} \left(\frac{a}{q} - \frac{\delta_0 r}{2qx}, \frac{a}{q} + \frac{\delta_0 r}{2qx}\right) \cup \bigcup_{\substack{q \le 2r\\ q \text{ even}}} \bigcup_{\substack{a \bmod q\\ (a,q)=1}} \left(\frac{a}{q} - \frac{\delta_0 r}{qx}, \frac{a}{q} + \frac{\delta_0 r}{qx}\right)$$

and $\delta_0 > 0$, $r \ge 1$ are given.

In other words, our major arcs will be few (that is, a constant number) and narrow. While [LW02] used relatively narrow major arcs as well, their number, as in all previous proofs of Vinogradov's result, is not bounded by a constant. (In his proof of the five-primes theorem, [Tao] is able to take a single major arc around 0; this is not possible here.)

What we are about to see is the general framework of the major arcs. This is naturally the place where the overlap with the existing literature is largest. Two important differences can nevertheless be singled out.

- The most obvious one is the presence of smoothing. At this point, it improves and simplifies error terms, but it also means that we will later need estimates for exponential sums on major arcs, and not just at the middle of each major arc. (If there is smoothing, we cannot use summation by parts to reduce the problem of estimating sums to a problem of counting primes in arithmetic progressions, or weighted by characters.)

- Since our L-function estimates for exponential sums will give bounds that are better than the trivial one by only a constant – even if it is a rather large constant – we need to be especially careful when estimating error terms, finding cancellation when possible.

3.1. **Decomposition of $S_\eta(\alpha, x)$ by characters.** What follows is largely classical; compare to [HL23] or, say, [Dav67, §26]. The only difference from the literature lies in the treatment of n non-coprime to q.

Write $\tau(\chi, b)$ for the Gauss sum

$$\tau(\chi, b) = \sum_{a \bmod q} \chi(a) e(ab/q) \tag{3.6}$$

associated to a $b \in \mathbb{Z}/q\mathbb{Z}$ and a Dirichlet character χ with modulus q. We let $\tau(\chi) = \tau(\chi, 1)$. If $(b, q) = 1$, then $\tau(\chi, b) = \chi(b^{-1})\tau(\chi)$.

Recall that χ^* denotes the primitive character inducing a given Dirichlet character χ. Writing $\sum_{\chi \bmod q}$ for a sum over all characters χ of $(\mathbb{Z}/q\mathbb{Z})^*)$, we see that, for any $a_0 \in \mathbb{Z}/q\mathbb{Z}$,

(3.7)
$$\begin{aligned}\frac{1}{\phi(q)} \sum_{\chi \bmod q} \tau(\overline{\chi}, b)\chi^*(a_0) &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \overline{\chi(a)} e(ab/q) \chi^*(a_0) \\ &= \sum_{\substack{a \bmod q \\ (a,q)=1}} \frac{e(ab/q)}{\phi(q)} \sum_{\chi \bmod q} \chi^*(a^{-1}a_0) = \sum_{\substack{a \bmod q \\ (a,q)=1}} \frac{e(ab/q)}{\phi(q)} \sum_{\chi \bmod q'} \chi(a^{-1}a_0),\end{aligned}$$

where $q' = q/\gcd(q, a_0^\infty)$. Now, $\sum_{\chi \bmod q'} \chi(a^{-1}a_0) = 0$ unless $a = a_0$ (in which case $\sum_{\chi \bmod q'} \chi(a^{-1}a_0) = \phi(q')$). Thus, (3.7) equals

$$\begin{aligned}\frac{\phi(q')}{\phi(q)} \sum_{\substack{a \bmod q \\ (a,q)=1 \\ a \equiv a_0 \bmod q'}} e(ab/q) &= \frac{\phi(q')}{\phi(q)} \sum_{\substack{k \bmod q/q' \\ (k, q/q')=1}} e\left(\frac{(a_0 + kq')b}{q}\right) \\ &= \frac{\phi(q')}{\phi(q)} e\left(\frac{a_0 b}{q}\right) \sum_{\substack{k \bmod q/q' \\ (k,q/q')=1}} e\left(\frac{kb}{q/q'}\right) = \frac{\phi(q')}{\phi(q)} e\left(\frac{a_0 b}{q}\right) \mu(q/q')\end{aligned}$$

provided that $(b, q) = 1$. (We are evaluating a *Ramanujan sum* in the last step.) Hence, for $\alpha = a/q + \delta/x$, $q \leq x$, $(a, q) = 1$,

$$\frac{1}{\phi(q)} \sum_\chi \tau(\overline{\chi}, a) \sum_n \chi^*(n)\Lambda(n)e(\delta n/x)\eta(n/x)$$

equals

$$\sum_n \frac{\mu((q, n^\infty))}{\phi((q, n^\infty))} \Lambda(n) e(\alpha n) \eta(n/x).$$

Since $(a,q)=1$, $\tau(\overline{\chi},a)=\chi(a)\tau(\overline{\chi})$. The factor $\mu((q,n^\infty))/\phi((q,n^\infty))$ equals 1 when $(n,q)=1$; the absolute value of the factor is at most 1 for every n. Clearly

$$\sum_{\substack{n\\(n,q)\neq 1}} \Lambda(n)\eta\left(\frac{n}{x}\right) = \sum_{p|q}\log p\sum_{\alpha\geq 1}\eta\left(\frac{p^\alpha}{x}\right).$$

Recalling the definition (3.1) of $S_\eta(\alpha,x)$, we conclude that

(3.8)

$$S_\eta(\alpha,x)=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\chi(a)\tau(\overline{\chi})S_{\eta,\chi^*}\left(\frac{\delta}{x},x\right)+O^*\left(2\sum_{p|q}\log p\sum_{\alpha\geq 1}\eta\left(\frac{p^\alpha}{x}\right)\right),$$

where

$$S_{\eta,\chi}(\beta,x)=\sum_n \Lambda(n)\chi(n)e(\beta n)\eta(n/x). \tag{3.9}$$

Hence $S_{\eta_1}(\alpha,x)S_{\eta_2}(\alpha,x)S_{\eta_3}(\alpha,x)e(-N\alpha)$ equals

$$\begin{aligned}&\frac{1}{\phi(q)^3}\sum_{\chi_1}\sum_{\chi_2}\sum_{\chi_3}\tau(\overline{\chi_1})\tau(\overline{\chi_2})\tau(\overline{\chi_3})\chi_1(a)\chi_2(a)\chi_3(a)e(-Na/q)\\ &\qquad\cdot S_{\eta_1,\chi_1^*}(\delta/x,x)S_{\eta_2,\chi_2^*}(\delta/x,x)S_{\eta_3,\chi_3^*}(\delta/x,x)e(-\delta N/x)\end{aligned}\tag{3.10}$$

plus an error term of absolute value at most

$$2\sum_{j=1}^{3}\prod_{j'\neq j}|S_{\eta_{j'}}(\alpha,x)|\sum_{p|q}\log p\sum_{\alpha\geq 1}\eta_j\left(\frac{p^\alpha}{x}\right).\tag{3.11}$$

We will later see that the integral of (3.11) over S^1 is negligible – for our choices of η_j, it will, in fact, be of size $O(x(\log x)^A)$, A a constant. (In (3.10), we have reduced our problems to estimating $S_{\eta,\chi}(\delta/x,x)$ for χ *primitive*; a more obvious way of reaching the same goal would have multiplied made (3.11) worse by a factor of about $\sqrt{q}$. The error term $O(x(\log x)^A)$ should be compared to the main term, which will be of size about a constant times x^2.)

3.2. The integral over the major arcs.

We are to estimate the integral (3.4), where the major arcs $\mathfrak{M}_{\delta_0,r}$ are defined as in (3.5). We will use $\eta_1=\eta_2=\eta_+$, $\eta_3(t)=\eta_*(\varkappa t)$, where η_+ and η_* will be set later.

We can write

$$\begin{aligned}S_{\eta,\chi}(\delta/x,x)=S_\eta(\delta/x,x)&=\int_0^\infty \eta(t/x)e(\delta t/x)dt+O^*(\mathrm{err}_{\eta,\chi}(\delta,x))\cdot x\\ &=\widehat{\eta}(-\delta)\cdot x+O^*(\mathrm{err}_{\eta,\chi_T}(\delta,x))\cdot x\end{aligned}\tag{3.12}$$

for $\chi=\chi_T$ the trivial character, and

$$S_{\eta,\chi}(\delta/x)=O^*(\mathrm{err}_{\eta,\chi}(\delta,x))\cdot x \tag{3.13}$$

for χ primitive and non-trivial. The estimation of the error terms err will come later; let us focus on (a) obtaining the contribution of the main term, (b) using estimates on the error terms efficiently.

The main term: three principal characters. The main contribution will be given by the term in (3.10) with $\chi_1=\chi_2=\chi_3=\chi_0$, where χ_0 is the principal character mod q.

The sum $\tau(\chi_0, n)$ is a *Ramanujan sum*; as is well-known (see, e.g., [IK04, (3.2)]),

$$\tau(\chi_0, n) = \sum_{d|(q,n)} \mu(q/d)d. \tag{3.14}$$

This simplifies to $\mu(q/(q,n))\phi((q,n))$ for q square-free. The special case $n = 1$ gives us that $\tau(\chi_0) = \mu(q)$.

Thus, the term in (3.10) with $\chi_1 = \chi_2 = \chi_3 = \chi_0$ equals

$$\frac{e(-Na/q)}{\phi(q)^3}\mu(q)^3 S_{\eta_+,\chi_0^*}(\delta/x, x)^2 S_{\eta_*,\chi_0^*}(\delta/x, x)e(-\delta N/x), \tag{3.15}$$

where, of course, $S_{\eta,\chi_0^*}(\alpha, x) = S_\eta(\alpha, x)$ (since χ_0^* is the trivial character). Summing (3.15) for $\alpha = a/q + \delta/x$ and a going over all residues mod q coprime to q, we obtain

$$\frac{\mu\left(\frac{q}{(q,N)}\right)\phi((q,N))}{\phi(q)^3}\mu(q)^3 S_{\eta_+,\chi_0^*}(\delta/x, x)^2 S_{\eta_*,\chi_0^*}(\delta/x, x)e(-\delta N/x).$$

The integral of (3.15) over all of $\mathfrak{M} = \mathfrak{M}_{\delta_0,r}$ (see (3.5)) thus equals
(3.16)

$$\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\phi((q,N))}{\phi(q)^3}\mu(q)^2\mu((q,N)) \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} S^2_{\eta_+,\chi_0^*}(\alpha, x) S_{\eta_*,\chi_0^*}(\alpha, x)e(-\alpha N)d\alpha$$

$$+ \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\phi((q,N))}{\phi(q)^3}\mu(q)^2\mu((q,N)) \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} S^2_{\eta_+,\chi_0^*}(\alpha, x) S_{\eta_*,\chi_0^*}(\alpha, x)e(-\alpha N)d\alpha.$$

The main term in (3.16) is
(3.17)

$$x^3 \cdot \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\phi((q,N))}{\phi(q)^3}\mu(q)^2\mu((q,N)) \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} (\widehat{\eta_+}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x)e(-\alpha N)d\alpha$$

$$+x^3 \cdot \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\phi((q,N))}{\phi(q)^3}\mu(q)^2\mu((q,N)) \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} (\widehat{\eta_+}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x)e(-\alpha N)d\alpha.$$

We would like to complete both the sum and the integral. Before, we should say that we will want to be able to use smoothing functions η_+ whose Fourier transform are not easy to deal with directly. All we want to require is that there be a smoothing function $\eta_\circ$, easier to deal with, such that $\eta_\circ$ be close to η_+ in ℓ_2 norm.

Assume, then, that

$$|\eta_+ - \eta_\circ|_2 \leq \epsilon_0 |\eta_\circ|,$$

where $\eta_\circ$ is thrice differentiable outside finitely many points and satisfies $\eta_\circ^{(3)} \in L_1$. Then (3.17) equals

(3.18)
$$\begin{aligned}
&x^3 \cdot \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\phi((q,N))}{\phi(q)^3} \mu(q)^2 \mu((q,N)) \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) d\alpha\\
&+x^3 \cdot \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\phi((q,N))}{\phi(q)^3} \mu(q)^2 \mu((q,N)) \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) d\alpha.
\end{aligned}$$

plus

$$O^*\left(x^2 \cdot \sum_q \frac{\mu(q)^2}{\phi(q)^2} \int_{-\infty}^{\infty} |(\widehat{\eta_+}(-\alpha))^2 - (\widehat{\eta_\circ}(-\alpha))^2||\widehat{\eta_*}(-\alpha)| d\alpha\right). \tag{3.19}$$

Here (3.19) is bounded by $2.82643x^2$ (by (B.4)) times

$$\begin{aligned}
&|\widehat{\eta_*}(-\alpha)|_\infty \cdot \sqrt{\int_{-\infty}^{\infty} |\widehat{\eta_+}(-\alpha) - \widehat{\eta_\circ}(-\alpha)|^2 d\alpha \cdot \int_{-\infty}^{\infty} |\widehat{\eta_+}(-\alpha) + \widehat{\eta_\circ}(-\alpha)|^2 d\alpha}\\
&\qquad \leq |\eta_*|_1 \cdot |\widehat{\eta_+} - \widehat{\eta_\circ}|_2 |\widehat{\eta_+} + \widehat{\eta_\circ}|_2 = |\eta_*|_1 \cdot |\eta_+ - \eta_\circ|_2 |\eta_+ + \eta_\circ|_2\\
&\qquad \leq |\eta_*|_1 \cdot |\eta_+ - \eta_\circ|_2 (2|\eta_\circ|_2 + |\eta_+ - \eta_\circ|_2) = |\eta_*|_1 |\eta_\circ|_2^2 \cdot (2+\epsilon_0)\epsilon_0.
\end{aligned}$$

Now, (3.18) equals

(3.20)
$$\begin{aligned}
&x^3 \int_{-\infty}^{\infty} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) \sum_{\substack{\frac{q}{(q,2)} \leq \min\left(\frac{\delta_0 r}{2|\alpha| x}, r\right)\\ \mu(q)^2 = 1}} \frac{\phi((q,N))}{\phi(q)^3} \mu((q,N)) d\alpha\\
&= x^3 \int_{-\infty}^{\infty} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) d\alpha \cdot \left(\sum_{q\geq 1} \frac{\phi((q,N))}{\phi(q)^3} \mu(q)^2 \mu((q,N))\right)\\
&-x^3 \int_{-\infty}^{\infty} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) \sum_{\substack{\frac{q}{(q,2)} > \min\left(\frac{\delta_0 r}{2|\alpha| x}, r\right)\\ \mu(q)^2 = 1}} \frac{\phi((q,N))}{\phi(q)^3} \mu((q,N)) d\alpha.
\end{aligned}$$

The last line in (3.20) is bounded[4] by

$$x^2 |\widehat{\eta_*}|_\infty \int_{-\infty}^{\infty} |\widehat{\eta_\circ}(-\alpha)|^2 \sum_{\frac{q}{(q,2)} > \min\left(\frac{\delta_0 r}{2|\alpha|}, r\right)} \frac{\mu(q)^2}{\phi(q)^2} d\alpha. \tag{3.21}$$

By (2.1) (with $k=3$), (B.11) and (B.12), this is at most

$$\begin{aligned}
&x^2 |\eta_*|_1 \int_{-\delta_0/2}^{\delta_0/2} |\widehat{\eta_\circ}|_\infty^2 \frac{4.31004}{r} d\alpha + 2x^2 |\eta_*|_1 \int_{\delta_0/2}^{\infty} \left(\frac{|\eta_\circ^{(3)}|_1}{(2\pi\alpha)^3}\right)^2 \frac{8.62008|\alpha|}{\delta_0 r} d\alpha\\
&\qquad \leq |\eta_*|_1 \left(4.31004 \delta_0 |\eta_\circ|_1^2 + 0.00113 \frac{|\eta_\circ^{(3)}|_1^2}{\delta_0^5}\right) \frac{x^2}{r}.
\end{aligned}$$

[4]This is obviously crude, in that we are bounding $\phi((q,N))/\phi(q)$ by 1. We are doing so in order to avoid a potentially harmful dependence on N.

It is easy to see that

$$\sum_{q\geq 1} \frac{\phi((q,N))}{\phi(q)^3}\mu(q)^2\mu((q,N)) = \prod_{p|N}\left(1-\frac{1}{(p-1)^2}\right)\cdot\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right).$$

Expanding the integral implicit in the definition of $\widehat{f}$,

$$\begin{aligned}\int_{\infty}^{\infty} (\widehat{\eta_\circ}(-\alpha x))^2 \widehat{\eta_*}(-\alpha x) e(-\alpha N) d\alpha = \\ \frac{1}{x}\int_0^\infty\int_0^\infty \eta_\circ(t_1)\eta_\circ(t_2)\eta_*\left(\frac{N}{x}-(t_1+t_2)\right)dt_1dt_2.\end{aligned} \tag{3.22}$$

(This is standard. One rigorous way to obtain (3.22) is to approximate the integral over $\alpha \in (-\infty,\infty)$ by an integral with a smooth weight, at different scales; as the scale becomes broader, the Fourier transform of the weight approximates (as a distribution) the δ function. Apply Plancherel.)

Hence, (3.17) equals

$$\begin{aligned} x^2\cdot\int_0^\infty\int_0^\infty \eta_\circ(t_1)\eta_\circ(t_2)\eta_*\left(\frac{N}{x}-(t_1+t_2)\right)dt_1dt_2 \\ \cdot\prod_{p|N}\left(1-\frac{1}{(p-1)^2}\right)\cdot\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right).\end{aligned} \tag{3.23}$$

(the main term) plus

$$\left(2.82643|\eta_\circ|_2^2(2+\epsilon_0)\cdot\epsilon_0 + \frac{4.31004\delta_0|\eta_\circ|_1^2 + 0.00113\frac{|\eta_\circ^{(3)}|_1^2}{\delta_0^5}}{r}\right)|\eta_*|_1 x^2 \tag{3.24}$$

Here (3.23) is just as in the classical case [IK04, (19.10)], except for the fact that a factor of 1/2 has been replaced by a double integral. We will later see how to choose our smoothing functions (and x, in terms of N) so as to make the double integral as large as possible.

What remains to estimate is the contribution of all the terms of the form $\mathrm{err}_{\eta,\chi}(\delta,x)$ in (3.12) and (3.13). Let us first deal with another matter – bounding the ℓ_2 norm of $|S_\eta(\alpha,x)|^2$ over the major arcs.

The ℓ_2 norm. We can always bound the integral of $|S_\eta(\alpha,x)|^2$ on the whole circle by Plancherel. If we only want the integral on certain arcs, we use the bound in Prop. 7.2 (based on work by Ramaré). If these arcs are really the major arcs – that is, the arcs on which we have useful analytic estimates – then we can hope to get better bounds using L-functions. This will be useful both to estimate the error terms in this section and to make the use of Ramaré's bounds more efficient later.

By (3.8),

$$\begin{aligned}
&\sum_{\substack{a \bmod q\\ \gcd(a,q)=1}} \left|S_\eta\left(\frac{a}{q}+\frac{\delta}{x},\chi\right)\right|^2\\
&= \frac{1}{\phi(q)^2}\sum_\chi\sum_{\chi'} \tau(\overline{\chi})\overline{\tau(\overline{\chi'})}\left(\sum_{\substack{a \bmod q\\ \gcd(a,q)=1}} \chi(a)\overline{\chi'(a)}\right)\cdot S_{\eta,\chi^*}(\delta/x,x)\overline{S_{\eta,\chi'^*}(\delta/x,x)}\\
&+ O^*\left(2(1+\sqrt{q})(\log x)^2|\eta|_\infty \max_\alpha |S_\eta(\alpha,x)| + \left((1+\sqrt{q})(\log x)^2|\eta|_\infty\right)^2\right)\\
&= \frac{1}{\phi(q)}\sum_\chi |\tau(\overline{\chi})|^2 |S_{\eta,\chi^*}(\delta/x,x)|^2 + K_{q,1}(2|S_\eta(0,x)|+K_{q,1}),
\end{aligned}$$

where

$$K_{q,1} = (1+\sqrt{q})(\log x)^2|\eta|_\infty.$$

As is well-known (see, e.g., [IK04, Lem. 3.1])

$$\tau(\chi) = \mu\left(\frac{q}{q^*}\right)\chi^*\left(\frac{q}{q^*}\right)\tau(\chi^*),$$

where q^* is the modulus of χ^* (i.e., the conductor of χ), and

$$|\tau(\chi^*)| = \sqrt{q^*}.$$

Using the expressions (3.12) and (3.13), we obtain

$$\begin{aligned}
&\sum_{\substack{a \bmod q\\ (a,q)=1}} \left|S_\eta\left(\frac{a}{q}+\frac{\delta}{x},x\right)\right|^2 = \frac{\mu^2(q)}{\phi(q)}\left|\widehat{\eta}(-\delta)x + O^*\left(\mathrm{err}_{\eta,\chi_T}(\delta,x)\cdot x\right)\right|^2\\
&+ \frac{1}{\phi(q)}\left(\sum_{\chi\neq\chi_T}\mu^2\left(\frac{q}{q^*}\right)q^*\cdot O^*\left(|\,\mathrm{err}_{\eta,\chi}(\delta,x)|^2x^2\right)\right) + K_{q,1}(2|S_\eta(0,x)|+K_{q,1})\\
&= \frac{\mu^2(q)x^2}{\phi(q)}\left(|\widehat{\eta}(-\delta)|^2 + O^*\left(|\mathrm{err}_{\eta,\chi_T}(\delta,x)(2|\eta|_1 + \mathrm{err}_{\eta,\chi_T}(\delta,x))|\right)\right)\\
&+ O^*\left(q\max_{\chi\neq\chi_T}|\,\mathrm{err}_{\eta,\chi^*}(\delta,x)|^2x^2 + K_{q,2}x\right),
\end{aligned}$$

where $K_{q,2} = K_{q,1}(2|S_\eta(0,x)|/x + K_{q,1}/x)$.

Thus, the integral of $|S_\eta(\alpha,x)|^2$ over $\mathfrak{M}$ (see (3.5)) is

$$\begin{aligned}
&\sum_{\substack{q\leq r\\ q\text{ odd}}}\sum_{\substack{a \bmod q\\ (a,q)=1}}\int_{\frac{a}{q}-\frac{\delta_0 r}{2qx}}^{\frac{a}{q}+\frac{\delta_0 r}{2qx}} |S_\eta(\alpha,x)|^2\,d\alpha + \sum_{\substack{q\leq 2r\\ q\text{ even}}}\sum_{\substack{a \bmod q\\ (a,q)=1}}\int_{\frac{a}{q}-\frac{\delta_0 r}{qx}}^{\frac{a}{q}+\frac{\delta_0 r}{qx}} |S_\eta(\alpha,x)|^2\,d\alpha\\
&= \sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)x^2}{\phi(q)}\int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} |\widehat{\eta}(-\alpha x)|^2\,d\alpha + \sum_{\substack{q\leq 2r\\ q\text{ even}}}\frac{\mu^2(q)x^2}{\phi(q)}\int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} |\widehat{\eta}(-\alpha x)|^2\,d\alpha\\
&+ O^*\left(\sum_q \frac{\mu^2(q)x^2}{\phi(q)}\cdot\frac{\gcd(q,2)\delta_0 r}{qx}\left(ET_{\eta,\frac{\delta_0 r}{2}}(2|\eta|_1 + ET_{\eta,\frac{\delta_0 r}{2}})\right)\right)\\
&+ \sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\delta_0 r x}{q}\cdot O^*\left(q \max_{\substack{\chi \bmod q\\ \chi\neq\chi_T\\ |\delta|\leq \delta_0 r/2q}} |\operatorname{err}_{\eta,\chi^*}(\delta,x)|^2 + \frac{K_{q,2}}{x}\right)\\
&+ \sum_{\substack{q\leq 2r\\ q\text{ even}}}\frac{2\delta_0 r x}{q}\cdot O^*\left(q \max_{\substack{\chi \bmod q\\ \chi\neq\chi_T\\ |\delta|\leq \delta_0 r/q}} |\operatorname{err}_{\eta,\chi^*}(\delta,x)|^2 + \frac{K_{q,2}}{x}\right),
\end{aligned}\tag{3.25}$$

where

$$ET_{\eta,s} = \max_{|\delta|\leq s} |\operatorname{err}_{\eta,\chi_T}(\delta,x)|$$

and χ_T is the trivial character. If all we want is an upper bound, we can simply remark that

$$\begin{aligned}
&x\sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)}{\phi(q)}\int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} |\widehat{\eta}(-\alpha x)|^2\,d\alpha + x\sum_{\substack{q\leq 2r\\ q\text{ even}}}\frac{\mu^2(q)}{\phi(q)}\int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} |\widehat{\eta}(-\alpha x)|^2\,d\alpha\\
&\leq \left(\sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)}{\phi(q)} + \sum_{\substack{q\leq 2r\\ q\text{ even}}}\frac{\mu^2(q)}{\phi(q)}\right)|\widehat{\eta}|_2^2 = 2|\eta|_2^2\sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)}{\phi(q)}.
\end{aligned}$$

If we also need a lower bound, we proceed as follows.

Again, we will work with an approximation $\eta_\circ$ such that (a) $|\eta-\eta_\circ|_2$ is small, (b) $\eta_\circ$ is thrice differentiable outside finitely many points, (c) $\eta_\circ^{(3)}\in L_1$. By (B.6),

$$\begin{aligned}
&x\sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)}{\phi(q)}\int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} |\widehat{\eta}(-\alpha x)|^2\,d\alpha\\
&\quad= \sum_{\substack{q\leq r\\ q\text{ odd}}}\frac{\mu^2(q)}{\phi(q)}\int_{-\frac{\delta_0 r}{2q}}^{\frac{\delta_0 r}{2q}} |\widehat{\eta_\circ}(-\alpha)|^2\,d\alpha + O^*\left(\frac{1}{2}\log r + 0.85\right)\cdot(|\widehat{\eta_\circ}|_2^2 - |\widehat{\eta}|_2^2).
\end{aligned}$$

Also,

$$x \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\mu^2(q)}{\phi(q)} \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} |\widehat{\eta}(-\alpha x)|^2 \, d\alpha = x \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} |\widehat{\eta}(-\alpha x)|^2 \, d\alpha.$$

By (2.1) and Plancherel,

$$\begin{aligned}\int_{-\frac{\delta_0 r}{2q}}^{\frac{\delta_0 r}{2q}} |\widehat{\eta_\circ}(-\alpha)|^2 \, d\alpha &= \int_{-\infty}^{\infty} |\widehat{\eta_\circ}(-\alpha)|^2 \, d\alpha - O^*\left(2\int_{\frac{\delta_0 r}{2q}}^{\infty} \frac{|\eta_\circ^{(3)}|_1^2}{(2\pi\alpha)^6} d\alpha\right)\\ &= |\eta_\circ|_2^2 + O^*\left(\frac{|\eta_\circ^{(3)}|_1^2 q^5}{5\pi^6(\delta_0 r)^5}\right),\end{aligned}$$

Hence

$$\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \int_{-\frac{\delta_0 r}{2q}}^{\frac{\delta_0 r}{2q}} |\widehat{\eta_\circ}(-\alpha)|^2 \, d\alpha = |\eta_\circ|_2^2 \cdot \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} + O^*\left(\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \frac{|\eta_\circ^{(3)}|_1^2 q^5}{5\pi^6(\delta_0 r)^5}\right).$$

Using (B.13), we get that

$$\begin{aligned}\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \frac{|\eta_\circ^{(3)}|_1^2 q^5}{5\pi^6(\delta_0 r)^5} &\leq \frac{1}{r} \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q) q}{\phi(q)} \cdot \frac{|\eta_\circ^{(3)}|_1^2}{5\pi^6\delta_0^5}\\ &\leq \frac{|\eta_\circ^{(3)}|_1^2}{5\pi^6\delta_0^5} \cdot \left(0.64787 + \frac{\log r}{4r} + \frac{0.425}{r}\right).\end{aligned}$$

Going back to (3.25), we use (B.2) to bound

$$\sum_q \frac{\mu^2(q) x^2}{\phi(q)} \frac{\gcd(q,2)\delta_0 r}{qx} \leq 2.59147 \cdot \delta_0 r x.$$

We also note that

$$\begin{aligned}\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{1}{q} + \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{2}{q} &= \sum_{q\leq r} \frac{1}{q} - \sum_{q\leq \frac{r}{2}} \frac{1}{2q} + \sum_{q\leq r} \frac{1}{q}\\ &\leq 2\log er - \log\frac{r}{2} \leq \log 2e^2 r.\end{aligned}$$

We have proven the following result.

Lemma 3.1. *Let $\eta : [0,\infty) \to \mathbb{R}$ be in $L_1 \cap L_\infty$. Let $S_\eta(\alpha, x)$ be as in (3.1) and let $\mathfrak{M} = \mathfrak{M}_{\delta_0,r}$ be as in (3.5). Let $\eta_\circ : [0,\infty) \to \mathbb{R}$ be thrice differentiable outside finitely many points. Assume $\eta_\circ^{(3)} \in L_1$.*

Assume $r \geq 182$. Then

(3.26)

$$\begin{aligned}\int_{\mathfrak{M}} |S_\eta(\alpha,x)|^2 d\alpha = L_{r,\delta_0} x &+ O^*\left(5.19\delta_0 x r \left(ET_{\eta,\frac{\delta_0 r}{2}} \cdot \left(|\eta|_1 + \frac{ET_{\eta,\delta_0 r/2}}{2}\right)\right)\right)\\ &+ O^*\left(\delta_0 x r\left(2 + \frac{3\log r}{2}\right) \cdot E_{\eta,r,\delta_0}^2 + \delta_0 r (\log 2e^2 r) K_{r,2}\right),\end{aligned}$$

where

(3.27)
$$E_{\eta,r,\delta_0} = \max_{\substack{\chi \bmod q\\ q\leq r\cdot \gcd(q,2)\\ |\delta|\leq \gcd(q,2)\delta_0 r/2q}} \sqrt{q}|\operatorname{err}_{\eta,\chi^*}(\delta,x)|, \qquad ET_{\eta,s} = \max_{|\delta|\leq s} |\operatorname{err}_{\eta,\chi_T}(\delta,x)|,$$
$$K_{r,2} = (1+\sqrt{2r})(\log x)^2|\eta|_\infty(2|S_\eta(0,x)|/x + (1+\sqrt{2r})(\log x)^2|\eta|_\infty/x)$$

and L_{r,δ_0} *satisfies both*

$$L_{r,\delta_0} \leq 2|\eta|_2^2 \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \tag{3.28}$$

and

$$\begin{aligned} L_{r,\delta_0} &= 2|\eta_\circ|_2^2 \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} + O^*(\log r + 1.7)\cdot(|\widehat{\eta_\circ}|_2^2 - |\widehat{\eta}|_2^2)\\ &+ O^*\left(\frac{2|\eta_\circ^{(3)}|_1^2}{5\pi^6\delta_0^5}\right)\cdot\left(0.64787 + \frac{\log r}{4r} + \frac{0.425}{r}\right). \end{aligned} \tag{3.29}$$

The error term $xrET_{\eta,\delta_0 r}$ will be very small, since it will be estimated using the Riemann zeta function; the error term involving $K_{r,2}$ will be completely negligible. The term involving $xr(r+1)E^2_{\eta,r,\delta_0}$; we see that it constrains us to have $|\operatorname{err}_{\eta,\chi}(x,N)|$ less than a constant times $1/r$ if we do not want the main term in the bound (3.26) to be overwhelmed.

3.2.1. *The triple product and its error terms.* There are at least two ways we can evaluate (3.4). One is to substitute (3.10) into (3.4). The disadvantages here are that (a) this can give rise to pages-long formulae, (b) this gives error terms proportional to $xr|\operatorname{err}_{\eta,\chi}(x,N)|$, meaning that, to win, we would have to show that $|\operatorname{err}_{\eta,\chi}(x,N)|$ is much smaller than $1/r$. What we will do instead is to use our ℓ_2 estimate (3.26) in order to bound the contribution of non-principal terms. This will give us a gain of almost $\sqrt{r}$ on the error terms; in other words, to win, it will be enough to show later that $|\operatorname{err}_{\eta,\chi}(x,N)|$ is much smaller than $1/\sqrt{r}$.

The contribution of the error terms in $S_{\eta_3}(\alpha,x)$ (that is, all terms involving the quantities $\operatorname{err}_{\eta,\chi}$ in expressions (3.12) and (3.13)) to (3.4) is

$$\begin{aligned} &\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{1}{\phi(q)} \sum_{\chi_3 \bmod q} \tau(\overline{\chi_3}) \sum_{\substack{a \bmod q\\ (a,q)=1}} \chi_3(a)e(-Na/q)\\ &\qquad \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} S_{\eta_+}(\alpha+a/q,x)^2 \operatorname{err}_{\eta_*,\chi_3^*}(\alpha x,x)e(-N\alpha)d\alpha\\ &+\sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{1}{\phi(q)} \sum_{\chi_3 \bmod q} \tau(\overline{\chi_3}) \sum_{\substack{a \bmod q\\ (a,q)=1}} \chi_3(a)e(-Na/q)\\ &\qquad \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} S_{\eta_+}(\alpha+a/q,x)^2 \operatorname{err}_{\eta_*,\chi_3^*}(\alpha x,x)e(-N\alpha)d\alpha. \end{aligned} \tag{3.30}$$

We should also remember the terms in (3.11); we can integrate them over all of $\mathbb{R}/\mathbb{Z}$, and obtain that they contribute at most

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}} & 2\sum_{j=1}^{3} \prod_{j'\neq j} |S_{\eta_{j'}}(\alpha,x)| \cdot \max_{q\leq r} \sum_{p|q} \log p \sum_{\alpha\geq 1} \eta_j\left(\frac{p^\alpha}{x}\right) d\alpha \\ &\leq 2\sum_{j=1}^{3} \prod_{j'\neq j} |S_{\eta_{j'}}(\alpha,x)|_2 \cdot \max_{q\leq r} \sum_{p|q} \log p \sum_{\alpha\geq 1} \eta_j\left(\frac{p^\alpha}{x}\right) \\ &= 2\sum_n \Lambda^2(n)\eta_+^2(n/x) \cdot \log r \cdot \max_{p\leq r} \sum_{\alpha\geq 1} \eta_*\left(\frac{p^\alpha}{x}\right) \\ &+ 4\sqrt{\sum_n \Lambda^2(n)\eta_+^2(n/x)\cdot \sum_n \Lambda^2(n)\eta_*^2(n/x)} \cdot \log r \cdot \max_{p\leq r}\sum_{\alpha\geq 1}\eta_*\left(\frac{p^\alpha}{x}\right)\end{aligned}$$

by Cauchy-Schwarz and Plancherel.

The absolute value of (3.30) is at most

$$\begin{aligned} &\sum_{\substack{q\leq r\\ q \text{ odd}}} \sum_{\substack{a \bmod q\\ (a,q)=1}} \sqrt{q} \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} \left|S_{\eta_+}(\alpha + a/q, x)\right|^2 d\alpha \cdot \max_{\substack{\chi \bmod q\\ |\delta|\leq \delta_0 r/2q}} |\mathrm{err}_{\eta_*,\chi^*}(\delta,x)| \\ &+ \sum_{\substack{q\leq 2r\\ q \text{ even}}} \sum_{\substack{a \bmod q\\ (a,q)=1}} \sqrt{q} \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} \left|S_{\eta_+}(\alpha + a/q, x)\right|^2 d\alpha \cdot \max_{\substack{\chi \bmod q\\ |\delta|\leq \delta_0 r/q}} |\mathrm{err}_{\eta_*,\chi^*}(\delta,x)| \\ &\qquad \leq \int_{\mathfrak{M}_{\delta_0,r}} \left|S_{\eta_+}(\alpha)\right|^2 d\alpha \cdot \max_{\substack{\chi \bmod q\\ q\leq r\cdot \gcd(q,2)\\ |\delta|\leq \gcd(q,2)\delta_0 r/q}} \sqrt{q}|\mathrm{err}_{\eta_*,\chi^*}(\delta,x)|. \end{aligned} \tag{3.31}$$

We can bound the integral of $|S_{\eta_+}(\alpha)|^2$ by (3.26).

What about the contribution of the error part of $S_{\eta_2}(\alpha,x)$? We can obviously proceed in the same way, except that, to avoid double-counting, $S_{\eta_3}(\alpha,x)$ needs to be replaced by

$$\frac{1}{\phi(q)}\tau(\overline{\chi_0})\widehat{\eta_3}(-\delta)\cdot x = \frac{\mu(q)}{\phi(q)}\widehat{\eta_3}(-\delta)\cdot x, \tag{3.32}$$

which is its main term (coming from (3.12)). Instead of having an ℓ_2 norm as in (3.31), we have the square-root of a product of two squares of ℓ_2 norms (by Cauchy-Schwarz), namely, $\int_{\mathfrak{M}} |S_{\eta_+}^*(\alpha)|^2 d\alpha$ and

$$\begin{aligned}\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)^2} \int_{-\frac{\delta_0 r}{2qx}}^{\frac{\delta_0 r}{2qx}} |\widehat{\eta_*}(-\alpha x)x|^2 d\alpha + \sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\mu^2(q)}{\phi(q)^2} \int_{-\frac{\delta_0 r}{qx}}^{\frac{\delta_0 r}{qx}} |\widehat{\eta_*}(-\alpha x)x|^2 d\alpha \\ \leq x|\widehat{\eta_*}|_2^2 \cdot \sum_q \frac{\mu^2(q)}{\phi(q)^2}.\end{aligned} \tag{3.33}$$

By (B.4), the sum over q is at most 2.82643.

As for the contribution of the error part of $S_{\eta_1}(\alpha,x)$, we bound it in the same way, using solely the ℓ_2 norm in (3.33) (and replacing both $S_{\eta_2}(\alpha,x)$ and $S_{\eta_3}(\alpha,x)$ by expressions as in (3.32)).

The total of the error terms is thus

$$\begin{aligned} & x \cdot \max_{\substack{\chi \bmod q \\ q \le r\cdot\gcd(q,2) \\ |\delta|\le \gcd(q,2)\delta_0 r/q}} \sqrt{q}\cdot |\mathrm{err}_{\eta_*,\chi^*}(\delta,x)| \cdot A \\ & + x \cdot \max_{\substack{\chi \bmod q \\ q \le r\cdot\gcd(q,2) \\ |\delta|\le \gcd(q,2)\delta_0 r/q}} \sqrt{q}\cdot |\mathrm{err}_{\eta_+,\chi^*}(\delta,x)| (\sqrt{A}+\sqrt{B_+})\sqrt{B_*}, \end{aligned} \tag{3.34}$$

where $A = (1/x)\int_{\mathfrak{M}} |S_{\eta_+}(\alpha,x)|^2 d\alpha$ (bounded as in (3.26)) and

$$B_* = 2.82643|\eta_*|_2^2, \qquad B_+ = 2.82643|\eta_+|_2^2. \tag{3.35}$$

In conclusion, we have proven

Proposition 3.2. *Let $x \ge 1$. Let $\eta_+, \eta_* : [0,\infty) \to \mathbb{R}$. Assume $\eta_+ \in C^2$, $\eta_+'' \in L_2$ and $\eta_+, \eta_* \in L^1 \cap L^2$. Let $\eta_\circ : [0,\infty) \to \mathbb{R}$ be thrice differentiable outside finitely many points. Assume $\eta_\circ^{(3)} \in L_1$ and $|\eta_+ - \eta_\circ|_2 < \epsilon_0 |\eta_\circ|_2$, where $\epsilon_0 \ge 0$.*

Let $S_\eta(\alpha,x) = \sum_n \Lambda(n) e(\alpha n)\eta(n/x)$. Let $\mathrm{err}_{\eta,\chi}$, χ primitive, be given as in (3.12) and (3.13). Let $\delta_0 > 0$, $r \ge 1$. Let $\mathfrak{M} = \mathfrak{M}_{\delta_0,r}$ be as in (3.5).

Then, for any $N \ge 0$,

$$\int_{\mathfrak{M}} S_{\eta_+}(\alpha,x)^2 S_{\eta_*}(\alpha,x) e(-N\alpha) d\alpha$$

equals

$$\begin{aligned} & C_0 C_{\eta_\circ,\eta_*} x^2 + \left(2.82643|\eta_\circ|_2^2(2+\epsilon_0)\cdot\epsilon_0 + \frac{4.31004\delta_0|\eta_\circ|_1^2 + 0.0012\frac{|\eta_\circ^{(3)}|_1^2}{\delta_0^5}}{r}\right)|\eta_*|_1 x^2 \\ & + O^*(E_{\eta_*,r,\delta_0} A_{\eta_+} + E_{\eta_+,r,\delta_0}\cdot 1.6812(\sqrt{A_{\eta_+}} + 1.6812|\eta_+|_2)|\eta_*|_2)\cdot x^2 \\ & + O^*\left(2 Z_{\eta_+^2,2}(x) LS_{\eta_*}(x,r)\cdot x + 4\sqrt{Z_{\eta_+^2,2}(x) Z_{\eta_*^2,2}(x)} LS_{\eta_+}(x,r)\cdot x\right), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} C_0 &= \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\cdot\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right), \\ C_{\eta_\circ,\eta_*} &= \int_0^\infty\int_0^\infty \eta_\circ(t_1)\eta_\circ(t_2)\eta_*\left(\frac{N}{x} - (t_1+t_2)\right) dt_1 dt_2, \end{aligned} \tag{3.37}$$

(3.38)

$$E_{\eta,r,\delta_0} = \max_{\substack{\chi \bmod q \\ q\leq \gcd(q,2)\cdot r \\ |\delta|\leq \gcd(q,2)\delta_0 r/2q}} \sqrt{q}\cdot |\operatorname{err}_{\eta,\chi^*}(\delta,x)|, \qquad ET_{\eta,s} = \max_{|\delta|\leq s/q} |\operatorname{err}_{\eta,\chi_T}(\delta,x)|,$$

$$A_\eta = L_{\eta,r,\delta_0}|\eta|_2^2 + 5.19\delta_0 r\left(ET_{\eta,\frac{\delta_0 r}{2}}\cdot\left(|\eta_1| + \frac{ET_{\eta,\delta_0 r/2}}{2}\right)\right)$$

$$+ \delta_0 r\left(2+\frac{3\log r}{2}\right)E^2_{\eta,r,\delta_0} + \delta_0 r x^{-1}(\log 2e^2 r)K_{r,2}$$

$$L_{\eta,r,\delta_0} \leq 2|\eta|_2^2 \sum_{\substack{q\leq r\\ q\ odd}} \frac{\mu^2(q)}{\phi(q)}$$

$$K_{r,2} = (1+\sqrt{2r})(\log x)^2|\eta|_\infty(2Z_{\eta,1}(x)/x + (1+\sqrt{2r})(\log x)^2|\eta|_\infty/x),$$

$$Z_{\eta,k}(x) = \frac{1}{x}\sum_n \Lambda^k(n)\eta(n/x), \qquad LS_\eta(x,r) = \log r\cdot \max_{p\leq r}\sum_{\alpha\geq 1}\eta\left(\frac{p^\alpha}{x}\right),$$

and $\operatorname{err}_{\eta,\chi}$ *is as in (3.12) and (3.13).*

Here is how to read these expressions. The error term in the first line of (3.36) will be small provided that ϵ_0 is small and r is large. The third line of (3.36) will be negligible, as will be the term $2\delta_0 r(\log er)K_{r,2}$ in the definition of A_η. (Clearly, $Z_{\eta,k}(x) \ll_\eta (\log x)^{k-1}$ and $LS_\eta(x,q) \ll_\eta \tau(q)\log x$ for any η of rapid decay.)

It remains to estimate the second line of (3.36), and this includes estimating A_η. We see that we will have to give very good bounds for E_{η,r,δ_0} when $\eta=\eta_+$ or $\eta=\eta_*$. (The same goes for $ET_{\eta_+,r\delta_0}$, for which the same method will work (and give even better bounds).) We also see that we want to make $C_0 C_{\eta_+,\eta_*}x^2$ as large as possible; it will be competing not just with the error terms here, but, more importantly, with the bounds from the minor arcs, which will be proportional to $|\eta_+|_2^2|\eta_*|_1$.

4. Optimizing and coordinating smoothing functions

One of our goals is to maximize the quantity $C_{\eta_\circ,\eta_*}$ in (3.37) relative to $|\eta_\circ|_2^2|\eta_*|_1$. One way to do this is to ensure that (a) η_* is concentrated on a very short[5] interval $[0,\epsilon)$, (b) $\eta_\circ$ is supported on the interval $[0,2]$, and is symmetric around $t=1$, meaning that $\eta_\circ(t)\sim\eta_\circ(2-t)$. Then, for $x\sim N/2$, the integral

$$\int_0^\infty\int_0^\infty \eta_\circ(t_1)\eta_\circ(t_2)\eta_*\left(\frac{N}{x}-(t_1+t_2)\right)dt_1dt_2$$

in (3.37) should be approximately equal to

$$|\eta_*|_1\cdot\int_0^\infty \eta_\circ(t)\eta_\circ\left(\frac{N}{x}-t\right)dt = |\eta_*|_1\cdot\int_0^\infty \eta_\circ(t)^2 dt = |\eta_*|_1\cdot|\eta_\circ|_2^2, \tag{4.1}$$

provided that $\eta_0(t)\geq 0$ for all t. It is easy to check (using Cauchy-Schwarz in the second step) that this is essentially optimal. (We will redo this rigorously in a little while.)

At the same time, the fact is that major-arc estimates are best for smoothing functions η of a particular form, and we have minor-arc estimates from [Hel] for

[5]This is an idea due to Bourgain in a related context [Bou99].

a different specific smoothing η_2. The issue, then, is how do we choose $\eta_\circ$ and η_* as above so that we can

- η_* is concentrated on $[0,\epsilon)$,
- $\eta_\circ$ is supported on $[0,2]$ and symmetric around $t=1$,
- we can give minor-arc and major-arc estimates for η_*,
- we can give major-arc estimates for a function η_+ close to $\eta_\circ$ in ℓ_2 norm?

4.1. **The symmetric smoothing function $\eta_\circ$.** We will later work with a smoothing function $\eta_\heartsuit$ whose Mellin transform decreases very rapidly. Because of this rapid decay, we will be able to give strong results based on an explicit formula for $\eta_\heartsuit$. The issue is how to define $\eta_\circ$, given $\eta_\heartsuit$, so that $\eta_\circ$ is symmetric around $t=1$ (i.e., $\eta_\circ(2-x)\sim\eta_\circ(x)$) and is very small for $x>2$.

We will later set $\eta_\heartsuit(t)=e^{-t^2/2}$. Let

$$h: t\mapsto \begin{cases} t^3(2-t)^3 e^{t/2-1/2} & \text{if } t\in[0,2],\\ 0 & \text{otherwise}\end{cases} \tag{4.2}$$

We define $\eta_\circ:\mathbb{R}\to\mathbb{R}$ by

$$\eta_\circ(t) = h(t)\eta_\heartsuit(t) = \begin{cases} t^3(2-t)^3 e^{-(t-1)^2/2} & \text{if } t\in[0,2],\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

It is clear that $\eta_\circ$ is symmetric around $t=1$ for $t\in[0,2]$.

4.1.1. *The product $\eta_\circ(t)\eta_\circ(\rho-t)$.* We now should go back and redo rigorously what we discussed informally around (4.1). More precisely, we wish to estimate

$$\eta_\circ(\rho) = \int_{-\infty}^{\infty}\eta_\circ(t)\eta_\circ(\rho-t)dt = \int_{-\infty}^{\infty}\eta_\circ(t)\eta_\circ(2-\rho+t)dt \tag{4.4}$$

for $\rho\le 2$ close to 2. In this, it will be useful that the Cauchy-Schwarz inequality degrades slowly, in the following sense.

Lemma 4.1. *Let V be a real vector space with an inner product $\langle\cdot,\cdot\rangle$. Then, for any $v,w\in V$ with $|w-v|_2\le|v|_2/2$,*

$$\langle v,w\rangle = |v|_2|w|_2 + O^*(2.71|v-w|_2^2).$$

Proof. By a truncated Taylor expansion,

$$\begin{aligned}\sqrt{1+x} &= 1+\frac{x}{2}+\frac{x^2}{2}\max_{0\le t\le 1}\frac{1}{4(1-(tx)^2)^{3/2}}\\ &= 1+\frac{x}{2}+O^*\left(\frac{x^2}{2^{3/2}}\right)\end{aligned}$$

for $|x|\le 1/2$. Hence, for $\delta=|w-v|_2/|v|_2$,

$$\begin{aligned}\frac{|w|_2}{|v|_2} &= \sqrt{1+\frac{2\langle w-v,v\rangle+|w-v|_2^2}{|v|_2^2}} = 1+\frac{2\frac{\langle w-v,v\rangle}{|v|_2^2}+\delta^2}{2}+O^*\left(\frac{(2\delta+\delta^2)^2}{2^{3/2}}\right)\\ &= 1+\delta+O^*\left(\left(\frac{1}{2}+\frac{(5/2)^2}{2^{3/2}}\right)\delta^2\right) = 1+\frac{\langle w-v,v\rangle}{|v|_2^2}+O^*\left(2.71\frac{|w-v|_2^2}{|v|_2^2}\right).\end{aligned}$$

Multiplying by $|v|_2^2$, we obtain that

$$|v|_2|w|_2 = |v|_2^2+\langle w-v,v\rangle+O^*\left(2.71|w-v|_2^2\right) = \langle v,w\rangle + O^*\left(2.71|w-v|_2^2\right).$$

□

Applying Lemma 4.1 to (4.4), we obtain that

$$
\begin{aligned}
(\eta_\circ * \eta_\circ)(\rho) &= \int_{-\infty}^{\infty} \eta_\circ(t)\eta_\circ((2-\rho)+t)dt \\
&= \sqrt{\int_{-\infty}^{\infty} |\eta_\circ(t)|^2 dt}\sqrt{\int_{-\infty}^{\infty} |\eta_\circ((2-\rho)+t)|^2 dt} \\
&+ O^*\left(2.71 \int_{-\infty}^{\infty} \left|\eta_\circ(t) - \eta_\circ((2-\rho)+t)\right|^2 dt\right) \\
&= |\eta_\circ|_2^2 + O^*\left(2.71 \int_{-\infty}^{\infty} \left(\int_0^{2-\rho} \left|\eta_\circ'(r+t)\right| dr\right)^2 dt\right) \\
&= |\eta_\circ|_2^2 + O^*\left(2.71(2-\rho) \int_0^{2-\rho} \int_{-\infty}^{\infty} \left|\eta_\circ'(r+t)\right|^2 dt dr\right) \\
&= |\eta_\circ|_2^2 + O^*(2.71(2-\rho)^2 |\eta_\circ'|_2^2).
\end{aligned} \tag{4.5}
$$

We will be working with η_* supported on the non-negative reals; we recall that $\eta_\circ$ is supported on $[0,2]$. Hence

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \eta_\circ(t_1)\eta_\circ(t_2)\eta_*\left(\frac{N}{x} - (t_1+t_2)\right) dt_1 dt_2 = \int_0^{\frac{N}{x}} (\eta_\circ * \eta_\circ)(\rho)\eta_*\left(\frac{N}{x} - \rho\right) d\rho \\
&\quad = \int_0^{\frac{N}{x}} (|\eta_\circ|_2^2 + O^*(2.71(2-\rho)^2|\eta_\circ'|_2^2)) \cdot \eta_*\left(\frac{N}{x} - \rho\right) d\rho \\
&\quad = |\eta_\circ|_2^2 \int_0^{\frac{N}{x}} \eta_*(\rho) d\rho + 2.71 |\eta_\circ'|_2^2 \cdot O^*\left(\int_0^{\frac{N}{x}} ((2-N/x)+\rho)^2 \eta_*(\rho) d\rho\right),
\end{aligned} \tag{4.6}
$$

provided that $N/x \geq 2$. We see that it will be wise to set N/x very slightly larger than 2. As we said before, η_* will be scaled so that it is concentrated on a small interval $[0,\epsilon)$.

4.2. **The approximation η_+ to $\eta_\circ$.** We will define $\eta_+ : [0,\infty) \to \mathbb{R}$ by

$$\eta_+(t) = h_H(t)\eta_\heartsuit(t), \tag{4.7}$$

where $h_H(t)$ is an approximation to $h(t)$ that is *band-limited* in the Mellin sense. *Band-limited* here means that the restriction of the Mellin transform to the imaginary axis has compact support $[-iH, iH]$, where $H > 0$ is a constant. Then, since

$$
\begin{aligned}
(M\eta_+)(s) = (M(h_H\eta_\heartsuit))(s) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} Mh_H(r) M\eta_\heartsuit(s-r)dr \\
&= \frac{1}{2\pi i}\int_{-iH}^{iH} Mh(r) M\eta_\heartsuit(s-r)dr
\end{aligned} \tag{4.8}
$$

(see (2.3)), the Mellin transform $M\eta_+$ will have decay properties similar to those of $M\eta_\heartsuit$ as $s \to \pm\infty$.

Let

$$I_H(s) = \begin{cases} 1 & \text{if } |\Im(s)| \leq H, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse Mellin transform of I_H is

$$(M^{-1}I_H)(y) = \frac{1}{2\pi i}\int_{-iH}^{iH} y^{-s} ds = \frac{1}{2\pi i}\frac{-y^{-s}}{\log y}|_{-iH}^{iH} = \frac{1}{\pi}\frac{\sin(H\log y)}{\log y}. \tag{4.9}$$

It is easy to check that the Mellin transform of this is indeed identical to $\chi_{[-iH,iH]}$ on the imaginary axis: (4.9) is the Dirichlet kernel under a change of variables. Now, in general, the Mellin transform of $f *_M g$ is $Mf \cdot Mg$. We define

$$\begin{aligned} h_H(t) = \left(h * \left(M^{-1}I_H\right)\right)(t) &= \int_{\frac{t}{2}}^{\infty} h(ty^{-1})\frac{\sin(H\log y)}{\pi\log y}\frac{dy}{y} \\ &= \int_0^{\frac{2}{t}} h(ty)\frac{\sin(H\log y)}{\pi\log y}\frac{dy}{y} \end{aligned} \tag{4.10}$$

and obtain that the Mellin transform of $h_H(t)$, on the imaginary axis, equals the restriction of Mh to the interval $[-iH, iH]$. We may adopt the convention that $h_H(t) = h(t) = 0$ for $t < 0$.

4.2.1. *The difference $\eta_+ - \eta_\circ$ in ℓ_2 norm.* By (4.3) and (4.7),

$$\begin{aligned} |\eta_+ - \eta_\circ|_2^2 &= \int_0^\infty |h_H(t)\eta_\heartsuit(t) - h(t)\eta_\heartsuit(t)|^2 dt \\ &\leq \max_{t\geq 0} |\eta_\heartsuit(t)|^2 t \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t}. \end{aligned} \tag{4.11}$$

Since the Mellin transform is an isometry (i.e., (2.4) holds),

$$\int_0^\infty |h_H(t) - h(t)|^2\frac{dt}{t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Mh_H(it) - Mh(it)|^2 dt = \frac{1}{\pi}\int_H^\infty |Mh(it)|^2 dt.$$

The maximum $\max_{t\geq 0} |\eta_\heartsuit(t)|^2 t$ is $1/\sqrt{2e}$.

Now, consider any $\phi : [0,\infty) \to \mathbb{C}$ that (a) has compact support (or fast decay), (b) satisfies $\phi(t) = O(t^3)$ near the origin and (c) is quadruply differentiable outside a finite set of points. We can show by integration by parts (cf. (2.1)) that, for $\Re(s) > -2$,

$$\begin{aligned} M\phi(s) &= \int_0^\infty \phi(x)x^s\frac{dx}{x} = -\int_0^\infty \phi'(x)\frac{x^s}{s}dx = \int_0^\infty \phi''(x)\frac{x^{s+1}}{s(s+1)}dx \\ &= -\int_0^\infty \phi^{(3)}(x)\frac{x^{s+2}}{s(s+1)(s+2)}dx = \lim_{t\to 0^+}\int_t^\infty \phi^{(4)}(x)\frac{x^{s+3}}{s(s+1)(s+2)(s+3)}dx, \end{aligned}$$

where $\phi^{(4)}(x)$ is understood in the sense of distributions at the finitely many places where it is not well-defined as a function.

Let $s = it$, $\phi = h$. Let $C_k = \int_0^\infty |h^{(k)}(x)|x^{k-1}dx$ for $0 \leq k \leq 4$. Then

$$Mh(it) = O^*\left(\frac{C_4}{|t||t+i||t+2i||t+3i|}\right). \tag{4.12}$$

Hence

$$\int_0^\infty |h_H(t) - h(t)|^2\frac{dt}{t} = \frac{1}{\pi}\int_H^\infty |Mh(it)|^2 dt \leq \frac{1}{\pi}\int_H^\infty \frac{C_4^2}{t^6}dt \leq \frac{C_4^2}{7\pi H^7} \tag{4.13}$$

and so

$$|\eta_+ - \eta_\circ|_2 \leq \frac{C_4}{\sqrt{7\pi}}\left(\frac{1}{2e}\right)^{1/4} \cdot \frac{1}{H^{7/2}}.$$

By (C.7), $C_4 = 3920.8817036284 + O(10^{-10})$. Thus

$$|\eta_+ - \eta_\circ|_2 \leq \frac{547.5562}{H^{7/2}}. \tag{4.14}$$

It will also be useful to bound

$$\left|\int_0^\infty (\eta_+(t) - \eta_\circ(t))^2 \log t \, dt\right|.$$

This is at most

$$\begin{aligned}\int_0^\infty (\eta_+(t) - \eta_\circ(t))^2 |\log t| dt &\leq \int_0^\infty |h_H(t)\eta_\heartsuit(t) - h(t)\eta_\heartsuit)|^2 |\log t| dt \\ &\leq \left(\max_{t\geq 0} |\eta_\heartsuit(t)|^2 \cdot t|\log t|\right) \cdot \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t}.\end{aligned}$$

Now

$$\max_{t\geq 0} |\eta_\heartsuit(t)|^2 t |\log t| = - \min_{t\in[0,1]} \eta_\heartsuit^2(t) t \log t \leq 0.3301223,$$

where we find the minimum by the bisection method (carried out rigorously, as in Appendix C.2, with 30 iterations), Hence, by (4.13),

$$\int_0^\infty (\eta_+(t) - \eta_\circ(t))^2 |\log t| dt \leq \frac{480.394}{H^{7/2}}. \tag{4.15}$$

* * *

As we said before, $(Mh_H)(it)$ is just the truncation of $(Mh)(it)$ to the interval $[-H, H]$. We can write down Mh explicitly:

$$Mh = e^{-1/2}(-1)^{-s}(8\gamma(s+3,-2) + 12\gamma(s+4,-2) + 6\gamma(s+5,-2) + \gamma(s+6,-2)),$$

where $\gamma(s,x)$ is the *(lower) incomplete Gamma function*

$$\gamma(s,x) = \int_0^x e^{-t} t^{s-1} dt.$$

However, it is easier to deal with Mh by means of bounds and approximations. Besides (4.12), note we have also derived

$$Mh(it) = O^*\left(\min\left(C_0, \frac{C_1}{|t|}, \frac{C_2}{|t||t+i|}, \frac{C_3}{|t||t+i||t+2i|}\right)\right). \tag{4.16}$$

By (C.2), (C.3), (C.4), (C.6) and (C.7),

$$\begin{aligned} &C_0 \leq 1.622284, \quad C_1 \leq 3.580004, \quad C_2 \leq 15.27957, \\ &C_3 \leq 131.3399, \quad C_4 \leq 3920.882. \end{aligned}$$

(We will compute rigorously far more precise bounds in Appendix C.1, but these bounds are all we could need.)

4.2.2. *Norms involving η_+.* Let us now bound some norms involving η_+. Relatively crude bounds will suffice in most cases.

First, by (4.14),

$$|\eta_+|_2 \leq |\eta_\circ|_2 + |\eta_+ - \eta_\circ|_2 \leq 0.80013 + \frac{547.5562}{H^{7/2}}, \tag{4.17}$$

where we obtain

$$|\eta_\circ|_2 = \sqrt{0.640205997\ldots} = 0.8001287\ldots \tag{4.18}$$

by symbolic integration. We can use the same idea to bound $|\eta_+(t)t^r|_2$, $r \geq -1/2$:

$$\begin{aligned}|\eta_+(t)t^r|_2 &\leq |\eta_\circ(t)t^r|_2 + |(\eta_+ - \eta_\circ)(t)t^r|_2\\ &\leq |\eta_\circ(t)t^r|_2 + \sqrt{\max_{t\geq 0} |\eta_\heartsuit(t)|^2 t^{2r+1} \cdot \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t}}\\ &\leq |\eta_\circ(t)t^r|_2 + \sqrt{\max_{t\geq 0} |\eta_\heartsuit(t)|^2 t^{2r+1}} \cdot \frac{C_{4,1}}{\sqrt{7\pi}} \frac{1}{H^{7/2}}.\end{aligned}$$

For example,

$$\begin{aligned}|\eta_\circ(t)t^{0.7}|_2 &\leq 0.80691, & \max_{t\geq 0} |\eta_\heartsuit(t)|^2 t^{2\cdot 0.7+1} &\leq 0.37486,\\ |\eta_\circ(t)t^{-0.3}|_2 &\leq 0.66168, & \max_{t\geq 0} |\eta_\heartsuit(t)|^2 t^{2\cdot(-0.3)+1} &\leq 0.5934.\end{aligned}$$

Thus

$$\begin{aligned}|\eta_+(t)t^{0.7}|_2 &\leq 0.80691 + \frac{511.92}{H^{7/2}},\\ |\eta_+(t)t^{-0.3}|_2 &\leq 0.66168 + \frac{644.08}{H^{7/2}}.\end{aligned} \tag{4.19}$$

The Mellin transform of η_+' equals $-(s-1)(M\eta_+)(s-1)$. Since the Mellin transform is an isometry in the sense of (2.4),

$$|\eta_+'|_2^2 = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left|M(\eta_+')(s)\right|^2 ds = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |s \cdot M\eta_+(s)|^2 \, ds.$$

Recall that $\eta_+(t) = h_H(t)\eta_\heartsuit(t)$. Thus, by (2.3), $M\eta_+(-1/2+it)$ equals $1/2\pi$ times the (additive) convolution of $Mh_H(it)$ and $M\eta_\heartsuit(-1/2+it)$. Therefore, for $s = -1/2+it$,

$$\begin{aligned}|s|\,|M\eta_+(s)| &= \frac{|s|}{2\pi} \int_{-H}^{H} Mh(ir) M\eta_\heartsuit(s-ir)dr\\ &\leq \frac{3}{2\pi} \int_{-H}^{H} |ir-1||Mh(ir)| \cdot |s-ir||M\eta_\heartsuit(s-ir)|dr\\ &= \frac{3}{2\pi}(f*g)(t),\end{aligned} \tag{4.20}$$

where $f(t) = |ir-1||Mh_H(it)|$ and $g(t) = |-1/2+it||M\eta_\heartsuit(-1/2+it)|$. (Since $(-1/2+i(t-r)) + (1+ir) = 1/2+it = s$, either either $|-1/2+i(t-r)| \geq |s|/3$ or $|1+ir| \geq 2|s|/3$; hence $|s-ir||ir-1| = |-1/2+i(t-r)||1+ir| \geq |s|/3$.) By Young's inequality (in a special case that follows from Cauchy-Schwarz), $|f*g|_2 \leq |f|_1|g|_2$. Again by Cauchy-Schwarz and Plancherel (i.e., isometry),

$$\begin{aligned}|f|_1^2 &= \left|\int_{-\infty}^{\infty} |ir-1||Mh_H(ir)|dr\right|^2 = \left|\int_{-H}^{H} |ir-1||Mh(ir)|dr\right|^2\\ &\leq 2H \int_{-H}^{H} |ir-1|^2 |Mh(ir)|^2 dr = 2H \int_{-H}^{H} |M((th)')(ir)|^2 dr\\ &= 4H\pi \int_0^\infty |(th)'(t)|^2 \frac{dt}{t} \leq 4H\pi \cdot 3.79234.\end{aligned}$$

Yet again by Plancherel,

$$|g|_2^2 = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |s|^2 |M\eta_\heartsuit(s)|^2 ds = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |(M(\eta_\heartsuit'))(s)|^2 ds = |\eta_\heartsuit'|_2^2 = \frac{\sqrt{\pi}}{4}.$$

Hence

$$(4.21)\quad |\eta_+'|_2 \leq \frac{1}{\sqrt{2\pi}}\cdot\frac{3}{2\pi}|f*g|_2 \leq \frac{3/2}{\sqrt{2}\pi^{3/2}}\cdot\sqrt{4H\pi}\cdot\frac{\pi^{1/4}}{2}\cdot\sqrt{3.79234} \leq 0.87531\sqrt{H}.$$

We can bound $|\eta_+' t^\sigma|_2$, $\sigma > 0$, in the same way. The only difference is that the integral is now taken on $(-1/2+\sigma-i\infty, -1/2+\sigma+i\infty)$ rather than $(-1/2-i\infty, -1/2+i\infty)$. For example, if $\sigma = 0.7$, then $|s||M\eta_+(s)| = (6/2\pi)(f*g)(t)$, where $f(t) = |ir-1||Mh_H(it)|$ and $g(t) = |0.2+it||M\eta_{\circ,1}(0.9+it)|$. Here

$$|g|_2^2 = \int_{1.2-i\infty}^{1.2-i\infty} |(M(\eta_\heartsuit'))(s)|^2 ds = |\eta_\heartsuit' \cdot t^{0.7}|_2^2 \leq 0.55091$$

and so

$$(4.22)\qquad |\eta_+' t^{0.7}|_2 \leq \frac{1}{\sqrt{2\pi}}\cdot\frac{3}{\pi}\cdot\sqrt{4H\pi}\cdot\sqrt{3.79234}\cdot\sqrt{0.55091} \leq 1.95201\sqrt{H}.$$

By isometry and (2.5),

$$|\eta_+\cdot\log|_2^2 = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |M(\eta_+\cdot\log)(s)|^2 ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |(M\eta_+)'(s)|^2 ds.$$

Now, $(M\eta_+)'(1/2+it)$ equals $1/2\pi$ times the additive convolution of $Mh_H(it)$ and $(M\eta_\heartsuit)'(1/2+it)$. Hence, by Young's inequality, $|(M\eta_+)'(1/2+it)|_2 \leq (1/2\pi)|Mh_H(it)|_1|(M\eta_\heartsuit)'(1/2+it)|_2$. By the definition of h_H, Cauchy-Schwarz and isometry,

$$\begin{aligned}|Mh_H(it)|_1 &= \int_{-H}^{H} |Mh(ir)|dr \leq \sqrt{2H}\sqrt{\int_{-H}^{H}|Mh(ir)|^2 dr}\\ &\leq \sqrt{2H}\sqrt{\int_{-\infty}^{\infty}|Mh(ir)|^2 dr} \leq \sqrt{2H}\sqrt{2\pi}|h(t)/\sqrt{t}|_2.\end{aligned}$$

Again by isometry and (2.5),

$$|(M\eta_\heartsuit)'(1/2+it)|_2 = \sqrt{2\pi}|\eta_\heartsuit\cdot\log|_2.$$

Hence

$$|\eta_+\cdot\log|_2 \leq \frac{2\pi}{(2\pi)^{3/2}}\sqrt{2H}|h(t)/\sqrt{t}|_2|\eta_\heartsuit\cdot\log|_2 = \frac{\sqrt{H}}{\sqrt{\pi}}|h(t)/\sqrt{t}|_2|\eta_\heartsuit\cdot\log|_2.$$

Since

$$(4.23)\qquad |h(t)/\sqrt{t}|_2 \leq 1.40927, \qquad |\eta_\heartsuit\cdot\log|_2 \leq 1.39554,$$

we get that

$$(4.24)\qquad |Mh_H(ir)|_1 \leq 1.99301\sqrt{2\pi H}$$

and

$$(4.25)\qquad |\eta_+\cdot\log|_2 \leq 1.10959\sqrt{H}.$$

Let us bound $|\eta_+(t)t^\sigma|_1$ for $\sigma \in (-1,\infty)$. By Cauchy-Schwarz and Plancherel,

$$\begin{aligned}|\eta_+(t)t^\sigma|_1 &= |\eta_\heartsuit(t)h_H(t)t^\sigma|_1 \leq |\eta_\heartsuit(t)t^{\sigma+1/2}|_2|h_H(t)/\sqrt{t}|_2\\ &= |\eta_\heartsuit(t)t^{\sigma+1/2}|_2\sqrt{\int_0^\infty |h_H(t)|^2\frac{dt}{t}}\\ &= |\eta_\heartsuit(t)t^{\sigma+1/2}|_2\cdot\sqrt{\frac{1}{2\pi}\int_{-i\infty}^{i\infty}|Mh_H(ir)|^2dr}\\ &\leq |\eta_\heartsuit(t)t^{\sigma+1/2}|_2\cdot\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|Mh(ir)|^2dr} \leq |\eta_\heartsuit(t)t^{\sigma+1/2}|_2\cdot|h(t)/\sqrt{t}|_2.\end{aligned}$$

Since

$$|\eta_\heartsuit t^{\sigma+1/2}|_2 = \sqrt{\int_0^\infty e^{-t^2}t^{2\sigma+1}dt} = \sqrt{\frac{\Gamma(\sigma+1)}{2}}$$

and $|h(t)/\sqrt{t}|_2$ is as in (4.23), we conclude that

$$|\eta_+(t)t^\sigma|_1 \leq 0.99651\cdot\sqrt{\Gamma(\sigma+1)}, \qquad |\eta_+|_1 \leq 0.996505. \tag{4.26}$$

Let us now get a bound for $|\eta_+|_\infty$. Recall that $\eta_+(t) = h_H(t)\eta_\heartsuit(t)$. Clearly

$$\begin{aligned}|\eta_+|_\infty &= |h_H(t)\eta_\heartsuit(t)|_\infty \leq |\eta_\circ|_\infty + |(h(t)-h_H(t))\eta_\heartsuit(t)|_\infty\\ &\leq |\eta_\circ|_\infty + \left|\frac{h(t)-h_H(t)}{t}\right|_\infty |\eta_\heartsuit(t)t|_\infty.\end{aligned} \tag{4.27}$$

Taking derivatives, we easily see that

$$|\eta_\circ|_\infty = \eta_\circ(1) = 1, \qquad |\eta_\heartsuit(t)t|_\infty = e^{-1/2}.$$

It remains to bound $|(h(t)-h_H(t))/t|_\infty$. By (4.10),

$$h_H(t) = \int_{\frac{t}{2}}^\infty h(ty^{-1})\frac{\sin(H\log y)}{\pi\log y}\frac{dy}{y} = \int_{-H\log\frac{2}{t}}^\infty h\left(\frac{t}{e^{w/H}}\right)\frac{\sin w}{\pi w}dw. \tag{4.28}$$

The *sine integral*

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt$$

is defined for all x; it tends to $\pi/2$ as $x\to+\infty$ and to $-\pi/2$ as $x\to-\infty$. We apply integration by parts to the second integral in (4.28), and obtain

$$\begin{aligned}h_H(t)-h(t) &= -\frac{1}{\pi}\int_{-H\log\frac{2}{t}}^\infty \left(\frac{d}{dw}h\left(\frac{t}{e^{w/H}}\right)\right)\mathrm{Si}(w)dw - h(t)\\ &= -\frac{1}{\pi}\int_0^\infty \left(\frac{d}{dw}h\left(\frac{t}{e^{w/H}}\right)\right)\left(\mathrm{Si}(w)-\frac{\pi}{2}\right)dw\\ &\quad -\frac{1}{\pi}\int_{-H\log\frac{2}{t}}^0 \left(\frac{d}{dw}h\left(\frac{t}{e^{w/H}}\right)\right)\left(\mathrm{Si}(w)+\frac{\pi}{2}\right)dw.\end{aligned}$$

Now

$$\left|\frac{d}{dw}h\left(\frac{t}{e^{w/H}}\right)\right| = \frac{te^{-w/H}}{H}\left|h'\left(\frac{t}{e^{w/H}}\right)\right| \leq \frac{t|h'|_\infty}{He^{w/H}}.$$

Integration by parts easily yields the bounds $|\operatorname{Si}(x) - \pi/2| < 2/x$ for $x > 0$ and $|\operatorname{Si}(x) + \pi/2| < 2/|x|$ for $x < 0$; we also know that $\operatorname{Si}(x) > 0$ for $x > 0$ and $\operatorname{Si}(x) < 0$ for $x < 0$. Hence

$$\begin{aligned}|h_H(t) - h(t)| &\le \frac{2t|h'|_\infty}{\pi H}\left(\int_0^1 \frac{\pi}{2} dw + \int_1^\infty \frac{2e^{-w/H}}{w} dw\right)\\ &= \frac{t|h'|_\infty}{H}\left(1 + \frac{4}{\pi}E_1(1/H)\right),\end{aligned}$$

where E_1 is the *exponential integral*

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt.$$

By [AS64],

$$0 < E_1(1/H) < \frac{\log(H+1)}{e^{1/H}} < \log H.$$

Hence

$$\frac{|h_H(t) - h(t)|}{t} < |h'|_\infty \cdot \frac{1 + \frac{4}{\pi}\log H}{H} \tag{4.29}$$

and so

$$|\eta_+|_\infty \le 1 + e^{-1/2}\left|\frac{h(t) - h_H(t)}{t}\right|_\infty < 1 + e^{-1/2}|h'|_\infty \cdot \frac{1 + \frac{4}{\pi}\log H}{H}.$$

The roots of $h''(t) = 0$ within $(0, 2)$ are $t = \sqrt{3} - 1$, $t = \sqrt{21} - 3$. A quick check gives that $|h'(\sqrt{21} - 3)| > |h'(\sqrt{3} - 1)|$. Hence

$$|h'|_\infty = |h'(\sqrt{21} - 3)| \le 3.65234. \tag{4.30}$$

We have proven

$$|\eta_+|_\infty < 1 + e^{-1/2} 3.65234 \frac{1 + \frac{4}{\pi}\log H}{H} < 1 + 2.21526 \frac{1 + \frac{4}{\pi}\log H}{H}. \tag{4.31}$$

We will need another bound of this kind, namely, for $\eta_+ \log t$. We start as in (4.27):

$$\begin{aligned}|\eta_+ \log t|_\infty &\le |\eta_\circ \log t|_\infty + |(h(t) - h_H(t))\eta_\heartsuit(t)\log t|_\infty\\ &\le |\eta_\circ \log t|_\infty + |(h - h_H(t))/t|_\infty |\eta_\heartsuit(t) t \log t|_\infty.\end{aligned}$$

By the bisection method with 30 iterations,

$$|\eta_\circ(t)\log t|_\infty \le 0.279491, \qquad |\eta_\heartsuit(t) t \log t|_\infty \le 0.346491.$$

Hence, by (4.29) and (4.30),

$$|\eta_+ \log t|_\infty \le 0.2795 + 1.26551 \cdot \frac{1 + \frac{4}{\pi}\log H}{H}. \tag{4.32}$$

4.3. **The smoothing function η_*.** Here the challenge is to define a smoothing function η_* that is good both for minor-arc estimates and for major-arc estimates. The two regimes tend to favor different kinds of smoothing function. For minor-arc estimates, both [Tao] and [Hel] use

$$\eta_2(t) = 4\max(\log 2 - |\log 2t|, 0) = ((2I_{[1/2,1]}) *_M (2I_{[1/2,1]}))(t), \tag{4.33}$$

where $I_{[1/2,1]}(t)$ is 1 if $t \in [1/2, 1]$ and 0 otherwise. For major-arc estimates, we will use a function based on

$$\eta_\heartsuit = e^{-t^2/2}.$$

(For example, we can give major-arc estimates for η_+, which is "based" on $\eta_\heartsuit$ (by (4.7)). We will actually use here the function $t^2 e^{-t^2/2}$, whose Mellin transform is $M\eta_\heartsuit(s+2)$ (by, e.g., [BBO10, Table 11.1]).)

We will follow the simple expedient of convolving the two smoothing functions, one good for minor arcs, the other one for major arcs. In general, let $\varphi_1, \varphi_2 : [0,\infty) \to \mathbb{C}$. It is easy to use bounds on sums of the form

$$S_{f,\varphi_1}(x) = \sum_n f(n)\varphi_1(n/x) \tag{4.34}$$

to bound sums of the form $S_{f,\varphi_1 *_M \varphi_2}$:

$$\begin{aligned} S_{f,\varphi_1 *_M \varphi_2} &= \sum_n f(n)(\varphi_1 *_M \varphi_2)\left(\frac{n}{x}\right) \\ &= \int_0^\infty \sum_n f(n)\varphi_1\left(\frac{n}{wx}\right)\varphi_2(w)\frac{dw}{w} = \int_0^\infty S_{f,\varphi_1}(wx)\varphi_2(w)\frac{dw}{w}. \end{aligned} \tag{4.35}$$

The same holds, of course, if φ_1 and φ_2 are switched, since $\varphi_1 *_M \varphi_2 = \varphi_2 *_M \varphi_1$. The only objection is that the bounds on (4.34) that we input might not be valid, or non-trivial, when the argument wx of $S_{f,\varphi_1}(wx)$ is very small. Because of this, it is important that the functions φ_1, φ_2 vanish at 0, and desirable that their first and second derivatives do so as well.

Let us see how this works out in practice for $\varphi_1 = \eta_2$. Here $\eta_2 : [0,\infty) \to \mathbb{R}$ is given by

$$\eta_2 = \eta_1 *_M \eta_1 = 4\max(\log 2 - |\log 2t|, 0), \tag{4.36}$$

where $\eta_1 = 2 \cdot I_{[1/2,1]}$. Bounding the sums $S_{\eta_2}(\alpha, x)$ on the minor arcs was the main subject of [Hel].

Before we use [Hel, Main Thm.], we need an easy lemma so as to simplify its statement.

Lemma 4.2. *For any* $q \geq 1$ *and any* $r \geq \max(3, q)$,

$$\frac{q}{\phi(q)} < \digamma(r),$$

where

$$\digamma(r) = e^\gamma \log\log r + \frac{2.50637}{\log\log r}. \tag{4.37}$$

Proof. Since $\digamma(r)$ is increasing for $r \geq 27$, the statement follows immediately for $q \geq 27$ by [RS62, Thm. 15]:

$$\frac{q}{\phi(q)} < \digamma(q) \leq \digamma(r).$$

For $r < 27$, it is clear that $q/\phi(q) \leq 2 \cdot 3/(1 \cdot 2) = 3$; it is also easy to see that $\digamma(r) > e^\gamma \cdot 2.50637 > 3$ for all $r > e$. □

It is time to quote the main theorem in [Hel]. Let $x \geq x_0$, $x_0 = 2.16 \cdot 10^{20}$. Let $2\alpha = a/q + \delta/x$, $q \leq Q$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/qQ$, where $Q = (3/4)x^{2/3}$. Then, if $3 \leq q \leq x^{1/3}/6$, [Hel, Main Thm.] gives us that

$$|S_{\eta_2}(\alpha, x)| \leq g_x\left(\max\left(1, \frac{|\delta|}{8}\right) \cdot q\right) x, \tag{4.38}$$

where

$$g_x(r) = \frac{(R_{x,2r}\log 2r + 0.5)\sqrt{F(r)} + 2.5}{\sqrt{2r}} + \frac{L_r}{r} + 3.2x^{-1/6}, \tag{4.39}$$

with

$$\begin{aligned} R_{x,t} &= 0.27125 \log\left(1 + \frac{\log 4t}{2\log\frac{9x^{1/3}}{2.004t}}\right) + 0.41415 \\ L_t &= F(t)\left(\log 2^{\frac{7}{4}} t^{\frac{13}{4}} + \frac{80}{9}\right) + \log 2^{\frac{16}{9}} t^{\frac{80}{9}} + \frac{111}{5}, \end{aligned} \tag{4.40}$$

(We are using Lemma 4.2 to bound all terms $1/\phi(q)$ appearing in [Hel, Main Thm.]; we are also using the obvious fact that, for $\delta_0 q$ fixed and $0 < a < b$, $\delta_0^a q^b$ is minimal when δ_0 is minimal.) If $q > x^{1/3}/6$, then, again by [Hel, Main Thm.],

$$|S_{\eta_2}(\alpha, x)| \leq h(x)x, \tag{4.41}$$

where

$$h(x) = 0.2727x^{-1/6}(\log x)^{3/2} + 1218x^{-1/3}\log x. \tag{4.42}$$

We will work with x varying within a range, and so we must pay some attention to the dependence of (4.38) and (4.41) on x.

Lemma 4.3. *Let $g_x(r)$ be as in (4.39) and $h(x)$ as in (4.42). Then*

$$x \mapsto \begin{cases} h(x) & \text{if } x < (6r)^3 \\ g_x(r) & \text{if } x \geq (6r)^3 \end{cases}$$

is a decreasing function of x for $r \geq 3$ fixed and $x \geq 21$.

Proof. It is clear from the definitions that $x \mapsto h(x)$ (for $x \geq 21$) and $x \mapsto g_{x,0}(r)$ are both decreasing. Thus, we simply have to show that $h(x_1) \geq g_{x_1,0}(r)$ for $x_1 = (6r)^3$. Since $x_1 \geq (6 \cdot 11)^3 > e^{12.5}$,

$$\begin{aligned} R_{x_1,2r} &\leq 0.27125 \log(0.065 \log x_1 + 1.056) + 0.41415 \\ &\leq 0.27125 \log((0.065 + 0.0845)\log x_1) + 0.41415 \leq 0.27215 \log\log x_1. \end{aligned}$$

Hence

$$\begin{aligned} R_{x_1,2r}\log 2r + 0.5 &\leq 0.27215 \log\log x_1 \log x_1^{1/3} - 0.27215 \log 12.5 \log 3 + 0.5 \\ &\leq 0.09072 \log\log x_1 \log x_1 - 0.255. \end{aligned}$$

At the same time,

$$\begin{aligned} F(r) &= e^\gamma \log\log \frac{x_1^{1/3}}{6} + \frac{2.50637}{\log\log r} \leq e^\gamma \log\log x_1 - e^\gamma \log 3 + 1.9521 \\ &\leq e^\gamma \log\log x_1 \end{aligned} \tag{4.43}$$

for $r \geq 37$, and we also get $F(r) \leq e^\gamma \log\log x_1$ for $r \in [11, 37]$ by the bisection method (carried out rigorously, as in Appendix C.2, with 10 iterations). Hence

$$\begin{aligned} &(R_{x_1,2r}\log 2r + 0.5)\sqrt{F(r)} + 2.5 \\ &\quad\leq (0.09072 \log\log x_1 \log x_1 - 0.255)\sqrt{e^\gamma \log\log x_1} + 2.5 \\ &\quad\leq 0.1211 \log x_1 (\log\log x_1)^{3/2} + 2, \end{aligned}$$

and so

$$\frac{(R_{x_1,2r}\log 2r + 0.5)\sqrt{F(r)} + 2.5}{\sqrt{2r}} \leq (0.21\log x_1 (\log\log x_1)^{3/2} + 3.47) x_1^{-1/6}.$$

Now, by (4.43),

$$\begin{aligned} L_r &\leq e^\gamma \log\log x_1 \cdot \left(\log 2^{\frac{7}{4}} (x_1^{1/3}/6)^{13/4} + \frac{80}{9}\right) + \log 2^{\frac{16}{9}} (x_1^{1/3}/6)^{\frac{80}{9}} + \frac{111}{5} \\ &\leq e^\gamma \log\log x_1 \cdot \left(\frac{13}{12}\log x_1 + 4.28\right) + \frac{80}{27}\log x + 7.51. \end{aligned}$$

It is clear that

$$\frac{4.28 e^\gamma \log\log x_1 + \frac{80}{27}\log x_1 + 7.51}{x_1^{1/3}/6} < 1218 x_1^{-1/3}\log x_1.$$

for $x_1 \geq e$.

It remains to show that

$$0.21\log x_1 (\log\log x_1)^{3/2} + 3.47 + 3.2 + \frac{13}{12} e^\gamma x_1^{-1/6}\log x_1 \log\log x_1 \tag{4.44}$$

is less than $0.2727(\log x_1)^{3/2}$ for x_1 large enough. Since $t \mapsto (\log t)^{3/2}/t^{1/2}$ is decreasing for $t > e^3$, we see that

$$\frac{0.21\log x_1 (\log\log x_1)^{3/2} + 6.67 + \frac{13}{12} e^\gamma x_1^{-1/6}\log x_1 \log\log x_1}{0.2727(\log x_1)^{3/2}} < 1$$

for all $x_1 \geq e^{34}$, simply because it is true for $x = e^{34} > e^{e^3}$.

We conclude that $h(x_1) \geq g_{x_1,0}(r) = g_{x_1,0}(x_1^{1/3}/6)$ for $x_1 \geq e^{34}$. We check that $h(x_1) \geq g_{x_1,0}(x_1^{1/3}/6)$ for all $x_1 \in [5832, e^{34}]$ as well by the bisection method (applied to $[5832, 583200]$ and to $[583200, e^{34}]$ with 30 iterations – in the latter interval, with 20 initial iterations). □

Lemma 4.4. *Let $R_{x,r}$ be as in (4.39). Then $t \to R_{e^t,r}(r)$ is convex-up for $t \geq 3\log 6r$.*

Proof. Since $t \to e^{-t/6}$ and $t \to t$ are clearly convex-up, all we have to do is to show that $t \to R_{e^t,r}$ is convex-up. In general, since

$$(\log f)'' = \left(\frac{f'}{f}\right)' = \frac{f'' f - (f')^2}{f^2},$$

a function of the form $(\log f)$ is convex-up exactly when $f''f - (f')^2 \geq 0$. If $f(t) = 1 + a/(t-b)$, we have $f''f - (f')^2 \geq 0$ whenever

$$(t + a - b)\cdot(2a) \geq a^2,$$

i.e., $a^2 + 2at \geq 2ab$, and that certainly happens when $t \geq b$. In our case, $b = 3\log(2.004r/9)$, and so $t \geq 3\log 6r$ implies $t \geq b$. □

Proposition 4.5. *Let $x \geq Kx_0$, $x_0 = 2.16\cdot 10^{20}$, $K \geq 1$. Let $S_\eta(\alpha, x)$ be as in (3.1). Let $\eta_* = \eta_2 *_M \varphi$, where η_2 is as in (4.36) and $\varphi : [0,\infty) \to [0,\infty)$ is continuous and in L^1.*

Let $2\alpha = a/q + \delta/x$, $q \leq Q$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/qQ$, where $Q = (3/4)x^{2/3}$. If $q \leq (x/K)^{1/3}/6$, then

$$S_{\eta_*}(\alpha, x) \leq g_{x,\varphi}\left(\max\left(1, \frac{|\delta|}{8}\right) q\right)\cdot |\varphi|_1 x, \tag{4.45}$$

where

$$g_{x,\varphi}(r) = \frac{(R_{x,K,\varphi,2r}\log 2r + 0.5)\sqrt{F(r)} + 2.5}{\sqrt{2r}} + \frac{L_r}{r} + 3.2K^{1/6}x^{-1/6},$$
$$R_{x,K,\varphi,t} = R_{x,t} + (R_{x/K,t} - R_{x,t})\frac{C_{\varphi,2}/|\varphi|_1}{\log K} \tag{4.46}$$

with $R_{x,t}$ *and* L_r *are as in (4.40), and*

$$C_{\varphi,2,K} = -\int_{1/K}^{1} \varphi(w)\log w\, dw. \tag{4.47}$$

If $q > (x/K)^{1/3}/6$*, then*

$$|S_{\eta_*}(\alpha,x)| \leq h_\varphi(x/K)\cdot|\varphi|_1 x,$$

where

$$h_\varphi(x) = h(x) + C_{\varphi,0,K}/|\varphi|_1,$$
$$C_{\varphi,0,K} = 1.04488\int_0^{1/K}|\varphi(w)|dw \tag{4.48}$$

and $h(x)$ *is as in (4.42).*

Proof. By (4.35),

$$S_{\eta_*}(\alpha,x) = \int_0^{1/K} S_{\eta_2}(\alpha,wx)\varphi(w)\frac{dw}{w} + \int_{1/K}^{\infty} S_{\eta_2}(\alpha,wx)\varphi(w)\frac{dw}{w}.$$

We bound the first integral by the trivial estimate $|S_{\eta_2}(\alpha,wx)| \leq |S_{\eta_2}(0,wx)|$ and Cor. A.3:

$$\begin{aligned}\int_0^{1/K}|S_{\eta_2}(0,wx)|\varphi(x)\frac{dw}{w} &\leq 1.04488\int_0^{1/K} wx\varphi(w)\frac{dw}{w}\\ &= 1.04488x\cdot\int_0^{1/K}\varphi(w)dw.\end{aligned}$$

If $w \geq 1/K$, then $wx \geq x_0$, and we can use (4.38) or (4.41). If $q > (x/K)^{1/3}/6$, then $|S_{\eta_2}(\alpha,wx)| \leq h(x/K)wx$ by (4.41); moreover, $|S_{\eta_2}(\alpha,y)| \leq h(y)y$ for $x/K \leq y < (6q)^3$ (by (4.41)) and $|S_{\eta_2}(\alpha,y)| \leq g_{y,1}(r)$ for $y \geq (6q)^3$ (by (4.38)). Thus, Lemma 4.3 gives us that

$$\begin{aligned}\int_{1/K}^{\infty}|S_{\eta_2}(\alpha,wx)|\varphi(w)\frac{dw}{w} &\leq \int_{1/K}^{\infty} h(x/K)wx\cdot\varphi(w)\frac{dw}{w}\\ &= h(x/K)x\int_{1/K}^{\infty}\varphi(w)dw \leq h(x/K)|\varphi|_1\cdot x.\end{aligned}$$

If $q \leq (x/K)^{1/3}/6$, we always use (4.38). We can use the coarse bound

$$\int_{1/K}^{\infty} 3.2x^{-1/6}\cdot wx\cdot\varphi(w)\frac{dw}{w} \leq 3.2K^{1/6}|\varphi|_1 x^{5/6}$$

Since L_r does not depend on x,

$$\int_{1/K}^{\infty}\frac{L_r}{r}\cdot wx\cdot\varphi(w)\frac{dw}{w} \leq \frac{L_r}{r}|\varphi|_1 x.$$

By Lemma 4.4 and $q \leq (x/K)^{1/3}/6$, $y \mapsto R_{e^y,t}$ is convex-up and decreasing for $y \in [\log(x/K), \infty)$. Hence

$$R_{wx,t} \leq \begin{cases} \frac{\log w}{\log \frac{1}{K}} R_{x/K,t} + \left(1 - \frac{\log w}{\log \frac{1}{K}}\right) R_{x,t} & \text{if } w < 1, \\ R_{x,t} & \text{if } w \geq 1. \end{cases}$$

Therefore

$$\begin{aligned} &\int_{1/K}^{\infty} R_{wx,t} \cdot wx \cdot \varphi(w) \frac{dw}{w} \\ &\quad \leq \int_{1/K}^{1} \left(\frac{\log w}{\log \frac{1}{K}} R_{x/K,t} + \left(1 - \frac{\log w}{\log \frac{1}{K}}\right) R_{x,t} \right) x\varphi(w) dw + \int_{1}^{\infty} R_{x,t}\varphi(w) x dw \\ &\quad \leq R_{x,t} x \cdot \int_{1/K}^{\infty} \varphi(w) dw + (R_{x/K,t} - R_{x,t}) \frac{x}{\log K} \int_{1/K}^{1} \varphi(w) \log w dw \\ &\quad \leq \left(R_{x,t} |\varphi|_1 + (R_{x/K,t} - R_{x,t}) \frac{C_{\varphi,2}}{\log K} \right) \cdot x, \end{aligned}$$

where

$$C_{\varphi,2,K} = -\int_{1/K}^{1} \varphi(w) \log w \, dw.$$

□

Lemma 4.6. *Let $x > K \cdot (6e)^3$, $K \geq 1$. Let $\eta_* = \eta_2 *_M \varphi$, where η_2 is as in (4.36) and $\varphi : [0, \infty) \to [0, \infty)$ is continuous and in L^1. Let $g_{x,\varphi}$ be as in (4.46).*

Then $g_{x,\varphi}(r)$ is a decreasing function of r for $r \geq 175$.

Proof. Taking derivatives, we can easily see that

$$r \mapsto \frac{\log \log r}{r}, \quad r \mapsto \frac{\log r}{r}, \quad r \mapsto \frac{(\log r)^2 \log \log r}{r} \tag{4.49}$$

are decreasing for $r \geq 20$. The same is true if $\log \log r$ is replaced by $\digamma(r)$, since $\digamma(r)/\log \log r$ is a decreasing function for $r \geq e$. Since $(C_{\varphi,2}/|\phi|_1)/\log K \leq 1$, we see that it is enough to prove that $r \mapsto R_{y,t} \log 2r \sqrt{\log \log r}/\sqrt{2r}$ is decreasing on r for $y = x$ and $y = x/K$ (under the assumption that $r \geq 175$).

Looking at (4.40) and at (4.49), it remains only to check that

$$r \mapsto \log \left(1 + \frac{\log 8r}{2 \log \frac{9x^{1/3}}{4.008r}}\right) \sqrt{\frac{\log \log r}{r}} \tag{4.50}$$

is decreasing on r for $r \geq 175$. Taking logarithms, and then derivatives, we see that we have to show that

$$\frac{\frac{\frac{1}{r}\ell + \frac{\log 8r}{r}}{2\ell^2}}{\left(1 + \frac{\log 8r}{2\ell}\right) \log \left(1 + \frac{\log 8r}{2\ell}\right)} + \frac{1}{2r \log r \log \log r} < \frac{1}{2r},$$

where $\ell = \log \frac{9x^{1/3}}{4.008r}$. Since $r \leq x^{1/3}/6$, $\ell \geq \log 54/4.008 > 2.6$. Thus, it is enough to ensure that

$$\frac{2/2.6}{\left(1 + \frac{\log 8r}{2\ell}\right) \log \left(1 + \frac{\log 8r}{2\ell}\right)} + \frac{1}{\log r \log \log r} < 1. \tag{4.51}$$

Since this is true for $r = 175$ and the left side is decreasing on r, the inequality is true for all $r \geq 175$.

□

Lemma 4.7. *Let* $x \geq 10^{24}$*. Let* $\phi : [0,\infty) \to [0,\infty)$ *be continuous and in* L^1*. Let* $g_{x,\phi}(r)$ *and* $h(x)$ *be as in (4.46) and (4.42), respectively. Then*

$$g_{x,\phi}\left(\frac{2}{3}x^{0.275}\right) \geq h(x/\log x).$$

Proof. We can bound $g_{x,\phi}(r)$ from below by

$$gm_x(r) = \frac{(R_{x,r}\log 2r + 0.5)\sqrt{F(r)} + 2.5}{\sqrt{2r}}$$

Let $r = (2/3)x^{0.275}$. Using the assumption that $x \geq 10^{24}$, we see that

$$R_{x,r} = 0.27125 \log\left(1 + \frac{\log\left(\frac{8}{3}x^{0.275}\right)}{2\log\left(\frac{9\cdot 3/2}{2.004}\cdot x^{\frac{1}{3}-0.275}\right)}\right) + 0.41415 \geq 0.67086.$$

Using $x \geq 10^{24}$ again, we get that

$$F(r) = e^{\gamma}\log\log r + \frac{2.50637}{\log\log r} \geq 5.72864.$$

Since $\log 2r = 0.275\log x + \log 4/3$, we conclude that

$$gm_x(r) \geq \frac{0.44156\log x + 4.1486}{\sqrt{4/3}\cdot x^{0.1375}}.$$

Recall that

$$h(x) = \frac{0.2727(\log x)^{3/2}}{x^{1/6}} + \frac{1218\log x}{x^{1/3}}.$$

It is easy to check that $(1/x^{0.1375})/((\log x)^{3/2}/x^{1/6})$ is increasing for $x \geq e^{51.43}$ (and hence for $x \geq 10^{24}$) and that $(1/x^{0.1375})/((\log x)/x^{1/3})$ is increasing for $x \geq e^{5.2}$ (and hence, again, for $x \geq 10^{24}$). Since

$$\frac{0.44156\log x + 2}{x^{0.1375}} > \frac{0.2727(\log x)^{3/2}}{x^{1/6}}, \qquad \frac{2.1486}{x^{0.1375}} > \frac{1218\log x}{x^{1/3}}$$

for $x \geq 10^{24}$, we are done. □

5. Mellin transforms and smoothing functions

5.1. **Exponential sums and L functions.** We must show how to estimate expressions of the form

$$S_{\eta,\chi}(\delta/x, x) = \sum_n \Lambda(n)\chi(n)e(\delta n/x)\eta(n/x),$$

where $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a smoothing function and δ is bounded by a large constant. We must also choose η. Let $f_\delta(t) = e(\delta t)\eta(t)$. By Mellin inversion,

$$\sum_{n=1}^{\infty}\Lambda(n)\chi(n)f_\delta(n/x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'(s,\chi)}{L(s,\chi)}F_\delta(s)x^s ds, \tag{5.1}$$

where F_δ is the Mellin transform of f_δ:

$$F_\delta = \int_0^\infty f_\delta(t)t^s\frac{dt}{t}. \tag{5.2}$$

The standard procedure here (already used in [HL23]) is to shift the line of integration in (5.1) to the left, picking up the contributions of the zeros of $L(s,\chi)$

along the way. (Once the line of integration is far enough to the left, the terms t^s within (5.1) become very small, and so the value of the integral ought to become very small, too.)

We will assume we know the zeros of $L(s,\chi)$ up to a certain height H_0 – meaning, in particular, that we know that non-trivial zeros with $\Im(s) \leq H_0$ lie on the critical line $\Re(s) = 1/2$ – but we have little control over the zeros above H_0. (The best zero-free regions available are not by themselves strong enough for our purposes.) Thus, we want to choose η so that the Mellin transform F_δ decays very rapidly – both for $\delta = 0$ and for δ non-zero and bounded.

5.2. **How to choose a smoothing function?** The method of *stationary phase* ([Olv74, §4.11], [Won01, §II.3])) suggests that the main contribution to (5.2) should come when the phase has derivative 0. The phase part of (5.2) is

$$e(\delta t) = t^{\Im(s)} = e^{2\pi i\delta t + \tau \log t}$$

(where we write $s = \sigma + i\tau$); clearly,

$$(2\pi\delta t + \tau \log t)' = 2\pi\delta + \frac{\tau}{t} = 0$$

when $t = -\tau/2\pi\delta$. This is meaningful when $t \geq 0$, i.e., $\mathrm{sgn}(\tau) \neq \mathrm{sgn}(\delta)$. The contribution of $t = -\tau/2\pi\delta$ to (5.2) is then

$$\eta(t)e(\delta t)t^{s-1} = \eta\left(\frac{-\tau}{2\pi\delta}\right) e^{-i\tau} \left(\frac{-\tau}{2\pi\delta}\right)^{\sigma+i\tau-1} \tag{5.3}$$

multiplied by a "width" approximately equal to a constant divided by

$$\sqrt{|(2\pi i\delta t + \tau \log t)''|} = \sqrt{|-\tau/t^2|} = \frac{2\pi|\delta|}{\sqrt{|\tau|}}.$$

The absolute value of (5.3) is

$$\eta\left(-\frac{\tau}{2\pi\delta}\right) \cdot \left|\frac{-\tau}{2\pi\delta}\right|^{\sigma-1}. \tag{5.4}$$

In other words, if $\mathrm{sgn}(\tau) \neq \mathrm{sgn}(\delta)$ and δ is not too small, asking that $F_\delta(\sigma+i\tau)$ decay rapidly as $|\tau| \to \infty$ amounts to asking that $\eta(t)$ decay rapidly as $t \to 0$. Thus, if we ask for $F_\delta(\sigma+i\tau)$ to decay rapidly as $|\tau| \to \infty$ for all moderate δ, we are requesting that

(1) $\eta(t)$ decay rapidly as $t \to \infty$,
(2) the Mellin transform $F_0(\sigma + i\tau)$ decay rapidly as $\tau \to \pm\infty$.

Requirement (2) is there because we also need to consider $F_\delta(\sigma + it)$ for δ very small, and, in particular, for $\delta = 0$.

There is clearly an uncertainty-principle issue here; one cannot do arbitrarily well in both aspects at the same time. Once we are conscious of this, the choice $\eta(t) = e^{-t}$ in Hardy-Littlewood actually looks fairly good: obviously, $\eta(t) = e^{-t}$ decays exponentially, and its Mellin transform $\Gamma(s+i\tau)$ also decays exponentially as $\tau \to \pm\infty$. Moreover, for this choice of η, the Mellin transform $F_\delta(s)$ can be written explicitly: $F_\delta(s) = \Gamma(s)/(1 - 2\pi i\delta)^s$.

It is not hard to work out an explicit formula[6] for $\eta(t) = e^{-t}$. However, it is not hard to see that, for $F_\delta(s)$ as above, $F_\delta(1/2+it)$ decays like $e^{-t/2\pi|\delta|}$, just as we expected from (5.4). This is a little too slow for our purposes: we will often have to work with relatively large δ, and we would like to have to check the zeroes of L functions only up to relatively low heights t. We will settle for a different choice of η: the Gaussian.

The decay of the Gaussian smoothing function $\eta(t) = e^{-t^2/2}$ is much faster than exponential. Its Mellin transform is $\Gamma(s/2)$, which decays exponentially as $\Im(s) \to \pm\infty$. Moreover, the Mellin transform $F_\delta(s)$ ($\delta \neq 0$), while not an elementary or very commonly occurring function, equals (after a change of variables) a relatively well-studied special function, namely, a parabolic cylinder function $U(a,z)$ (or, in Whittaker's [Whi03] notation, $D_{-a-1/2}(z)$).

For δ not too small, the main term will indeed work out to be proportional to $e^{-(\tau/2\pi\delta)^2/2}$, as the method of stationary phase indicated. This is, of course, much better than $e^{-\tau/2\pi|\delta|}$. The "cost" is that the Mellin transform $\Gamma(s/2)$ for $\delta = 0$ now decays like $e^{-(\pi/4)|\tau|}$ rather than $e^{-(\pi/2)|\tau|}$. This we can certainly afford – the main concern was $e^{-|\tau|/\delta}$ for $\delta \sim 10^5$, say.

5.3. **The Mellin transform of the twisted Gaussian.** We wish to approximate the Mellin transform

$$F_\delta(s) = \int_0^\infty e^{-t^2/2} e(\delta t) t^s \frac{dt}{t},$$

where $\delta \in \mathbb{R}$. The parabolic cylinder function $U : \mathbb{C}^2 \to \mathbb{C}$ is given by

$$U(a,z) = \frac{e^{-z^2/4}}{\Gamma\left(\frac{1}{2}+a\right)} \int_0^\infty t^{a-\frac{1}{2}} e^{-\frac{1}{2}t^2 - zt} dt$$

for $\Re(a) > -1/2$; this can be extended to all $a, z \in \mathbb{C}$ either by analytic continuation or by other integral representations ([AS64, §19.5], [Tem10, §12.5(i)]). Hence

$$F_\delta(s) = e^{(\pi i \delta)^2} \Gamma(s) U\left(s - \frac{1}{2}, -2\pi i \delta\right). \tag{5.5}$$

The second argument of U is purely imaginary; it would be otherwise if a Gaussian of non-zero mean were chosen.

Let us briefly discuss the state of knowledge up to date on Mellin transforms of "twisted" Gaussian smoothings, that is, $e^{-t^2/2}$ multiplied by an additive character $e(\delta t)$. As we have just seen, these Mellin transforms are precisely the parabolic cylinder functions $U(a,z)$.

The function $U(a,z)$ has been well-studied for a and z real; see, e.g., [Tem10]. Less attention has been paid to the more general case of a and z complex. The most notable exception is by far the work of Olver [Olv58], [Olv59], [Olv61], [Olv65]; he gave asymptotic series for $U(a,z)$, $a, z \in \mathbb{C}$. These were asymptotic series in the sense of Poincaré, and thus not in general convergent; they would solve our problem if and only if they came with error term bounds. Unfortunately,

[6]There may be a minor gap in the literature in this respect. The explicit formula given in [HL23, Lemma 4] does not make all constants explicit. The constants and trivial-zero terms were fully worked out for $q = 1$ by [Wig20] (cited in [MV07, Exercise 12.1.1.8(c)]; the sign of $\mathrm{hyp}_{\kappa,q}(z)$ there seems to be off). As was pointed out by Landau (see [Har66, p. 628]), [HL23, Lemma 4] actually has mistaken terms for χ non-primitive. (The author thanks R. C. Vaughan for this information and the references.)

it would seem that all fully explicit error terms in the literature are either for a and z real, or for a and z outside our range of interest (see both Olver's work and [TV03].) The bounds in [Olv61] involve non-explicit constants. Thus, we will have to find expressions with explicit error bounds ourselves. Our case is that of a in the critical strip, z purely imaginary.

Gaussian smoothing has been used before in number theory; see, notably, [HB79]. What is new here is that we will derive fully explicit bounds on the Mellin transform of the twisted Gaussian. This means that the Gaussian smoothing will be a real option in explicit work on exponential sums in number theory from now on.

5.3.1. *General approach and situation.* We will use the *saddle-point method* (see, e.g., [dB81, §5], [Olv74, §4.7], [Won01, §II.4]) to obtain bounds with an optimal leading-order term and small error terms. (We used the stationary-phase method solely as an exploratory tool.)

What do we expect to obtain? Both the asymptotic expressions in [Olv59] and the bounds in [Olv61] make clear that, if the sign of $\tau = \Im(s)$ is different from that of δ, there will a change in behavior when τ gets to be of size about $(2\pi\delta)^2$. This is unsurprising, given our discussion using stationary phase: for $|\Im(a)|$ smaller than a constant times $|\Im(z)|^2$, the term proportional to $e^{-(\pi/4)|\tau|} = e^{-|\Im(a)|/2}$ should be dominant, whereas for $|\Im(a)|$ much larger than a constant times $|\Im(z)|^2$, the term proportional to $e^{-\frac{1}{2}\left(\frac{\tau}{2\pi\delta}\right)^2}$ should be dominant.

5.3.2. *Setup.* We write

$$\phi(u) = \frac{u^2}{2} - (2\pi i\delta)u - i\tau \log u \tag{5.6}$$

for u real or complex, so that

$$F_\delta(s) = \int_0^\infty e^{-\phi(u)} u^\sigma \frac{du}{u}.$$

We will be able to shift the contour of integration as we wish, provided that it starts at 0 and ends at a point at infinity while keeping within the sector $\arg(u) \in (-\pi/4, \pi/4)$.

We wish to find a saddle point. At a saddle point, $\phi'(u) = 0$. This means that

$$u - 2\pi i\delta - \frac{i\tau}{u} = 0, \qquad \text{i.e.,} \qquad u^2 + i\ell u - i\tau = 0, \tag{5.7}$$

where $\ell = -2\pi\delta$. The solutions to $\phi'(u) = 0$ are thus

$$u_0 = \frac{-i\ell \pm \sqrt{-\ell^2 + 4i\tau}}{2}. \tag{5.8}$$

The second derivative at u_0 is

$$\phi''(u_0) = \frac{1}{u_0^2}\left(u_0^2 + i\tau\right) = \frac{1}{u_0^2}(-i\ell u_0 + 2i\tau). \tag{5.9}$$

Assign the names $u_{0,+}$, $u_{0,-}$ to the roots in (5.8) according to the sign in front of the square-root (where the square-root is defined so as to have argument in $(-\pi/2, \pi/2]$).

We assume without loss of generality that $\tau \geq 0$. We shall also assume at first that $\ell \geq 0$ (i.e., $\delta \leq 0$), as the case $\ell < 0$ is much easier.

5.3.3. *The saddle point.* Let us start by estimating

$$\left|u_{0,+}^s e^{y/2}\right| = |u_{0,+}|^\sigma e^{-\arg(u_{0,+})\tau} e^{y/2}, \tag{5.10}$$

where $y = \Re(-\ell i u_0)$. (This is the main part of the contribution of the saddle point, without the factor that depends on the contour.) We have

$$y = \Re\left(-\frac{i\ell}{2}\left(-i\ell + \sqrt{-\ell^2+4i\tau}\right)\right) = \Re\left(-\frac{\ell^2}{2} - \frac{i\ell^2}{2}\sqrt{-1+\frac{4i\tau}{\ell^2}}\right). \tag{5.11}$$

Solving a quadratic equation, we get that

$$\sqrt{-1+\frac{4i\tau}{\ell^2}} = \sqrt{\frac{j(\rho)-1}{2}} + i\sqrt{\frac{j(\rho)+1}{2}}, \tag{5.12}$$

where $j(\rho) = (1+\rho^2)^{1/2}$ and $\rho = 4\tau/\ell^2$. Thus

$$y = \frac{\ell^2}{2}\left(\sqrt{\frac{j(\rho)+1}{2}} - 1\right).$$

Let us now compute the argument of $u_{0,+}$:

$$\begin{aligned}
\arg(u_{0,+}) &= \arg\left(-i\ell + \sqrt{-\ell^2+4i\tau}\right) = \arg\left(-i + \sqrt{-1+i\rho}\right)\\
&= \arg\left(-i + \sqrt{\frac{-1+j(\rho)}{2}} + i\sqrt{\frac{1+j(\rho)}{2}}\right)\\
&= \arcsin\left(\frac{\sqrt{\frac{1+j(\rho)}{2}}-1}{\sqrt{2\sqrt{\frac{1+j(\rho)}{2}}\left(\sqrt{\frac{1+j(\rho)}{2}}-1\right)}}\right)\\
&= \arcsin\left(\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{2}{1+j(\rho)}}\right)}\right) = \frac{1}{2}\arccos\sqrt{\frac{2}{1+j(\rho)}}
\end{aligned} \tag{5.13}$$

(by $\cos 2\theta = 1 - 2\sin^2\theta$). Thus

$$\begin{aligned}
-\arg(u_{0,+})\tau + \frac{y}{2} &= -\left(\arccos\sqrt{\frac{2}{1+j(\rho)}} - \frac{\ell^2}{2\tau}\left(\sqrt{\frac{j(\rho)+1}{2}}-1\right)\right)\frac{\tau}{2}\\
&= -\left(\arccos\frac{1}{\upsilon(\rho)} - \frac{2(\upsilon(\rho)-1)}{\rho}\right)\frac{\tau}{2},
\end{aligned} \tag{5.14}$$

where $\upsilon(\rho) = \sqrt{(1+j(\rho))/2}$.

It is clear that

$$\lim_{\rho\to\infty}\left(\arccos\frac{1}{\upsilon(\rho)} - \frac{2(\upsilon(\rho)-1)}{\rho}\right) = \frac{\pi}{2} \tag{5.15}$$

whereas

$$\arccos\frac{1}{\upsilon(\rho)} - \frac{2(\upsilon(\rho)-1)}{\rho} \sim \frac{\rho}{2} - \frac{\rho}{4} = \frac{\rho}{4} \tag{5.16}$$

as $\rho \to 0^+$.

We are still missing a factor of $|u_{0,+}|^\sigma$ (from (5.10)), a factor of $|u_{0,+}|^{-1}$ (from the invariant differential du/u) and a factor of $1/\sqrt{|\phi''(u_{0,+})|}$ (from the passage by the saddle-point along a path of steepest descent). By (5.9), this is

$$\frac{|u_{0,+}|^{\sigma-1}}{\sqrt{|\phi''(u_{0,+})|}} = \frac{|u_{0,+}|^{\sigma-1}}{\frac{1}{|u_{0,+}|}\sqrt{|-i\ell u_{0,+}+2i\tau|}} = \frac{|u_{0,+}|^{\sigma}}{\sqrt{|-i\ell u_{0,+}+2i\tau|}}.$$

By (5.8) and (5.12),

$$\begin{aligned}|u_{0,+}| &= \left|\frac{-i\ell+\sqrt{-\ell^2+4i\tau}}{2}\right| = \frac{\ell}{2}\cdot\left|\sqrt{\frac{-1+j(\rho)}{2}}+\left(\sqrt{\frac{1+j(\rho)}{2}}-1\right)i\right|\\ &= \frac{\ell}{2}\sqrt{\frac{-1+j(\rho)}{2}+\frac{1+j(\rho)}{2}+1-2\sqrt{\frac{1+j(\rho)}{2}}}\\ &= \frac{\ell}{2}\sqrt{1+j(\rho)-2\sqrt{\frac{1+j(\rho)}{2}}} = \frac{\ell}{\sqrt{2}}\sqrt{\upsilon(\rho)^2-\upsilon(\rho)}.\end{aligned} \tag{5.17}$$

Proceeding as in (5.11), we obtain that

$$\begin{aligned}|-i\ell u_{0,+}+2i\tau| &= \left|-\frac{i\ell}{2}\left(-i\ell+\ell\sqrt{-1+\frac{4i\tau}{\ell^2}}\right)+2i\tau\right|\\ &= \left|-\frac{\ell^2}{2}+2i\tau+\frac{\ell^2}{2}\sqrt{\frac{j(\rho)+1}{2}}-\frac{i\ell^2}{2}\sqrt{\frac{j(\rho)-1}{2}}\right|\\ &= \frac{\ell^2}{2}\sqrt{\left(-1+\sqrt{\frac{j(\rho)+1}{2}}\right)^2+\left(\rho-\sqrt{\frac{j(\rho)-1}{2}}\right)^2}\\ &= \frac{\ell^2}{2}\sqrt{j(\rho)+\rho^2+1-2\sqrt{\frac{j(\rho)+1}{2}}-2\rho\sqrt{\frac{j(\rho)-1}{2}}}.\end{aligned} \tag{5.18}$$

Since $\sqrt{j(\rho)-1} = \rho/\sqrt{j(\rho)+1}$, this means that

$$\begin{aligned}|-i\ell u_{0,+}+2i\tau| &= \frac{\ell^2}{2}\sqrt{j(\rho)+\rho^2+1-\sqrt{\frac{2}{j(\rho)+1}}(j(\rho)+1+\rho^2)}\\ &= \frac{\ell^2}{2}\sqrt{j(\rho)+j(\rho)^2-(\upsilon(\rho))^{-1}(j(\rho)+j(\rho)^2)}\\ &= \frac{\ell^2}{2}\sqrt{2\upsilon(\rho)^2 j(\rho)(1-(\upsilon(\rho))^{-1})} = \frac{\ell^2\sqrt{j(\rho)}}{\sqrt{2}}\sqrt{\upsilon(\rho)^2-\upsilon(\rho)}.\end{aligned} \tag{5.19}$$

Hence

$$\begin{aligned}\frac{|u_{0,+}|^{\sigma}}{\sqrt{|-i\ell u_{0,+}+2i\tau|}} &= \frac{\left(\frac{\ell}{\sqrt{2}}\sqrt{\upsilon(\rho)^2-\upsilon(\rho)}\right)^{\sigma}}{\frac{\ell(j(\rho))^{1/4}}{2^{1/4}}(\upsilon(\rho)^2-\upsilon(\rho))^{1/4}} = \frac{\ell^{\sigma-1}}{2^{\frac{\sigma}{2}-\frac{1}{4}}j(\rho)^{\frac{1}{4}}}(\upsilon(\rho)^2-\upsilon(\rho))^{\frac{\sigma}{2}-\frac{1}{4}}\\ &= \frac{2^{\frac{\sigma}{2}-\frac{3}{4}}}{\rho^{\frac{\sigma-1}{2}}j(\rho)^{\frac{1}{4}}}(\upsilon(\rho)^2-\upsilon(\rho))^{\frac{\sigma}{2}-\frac{1}{4}}\cdot\tau^{\frac{\sigma-1}{2}}.\end{aligned}$$

It remains to determine the direction of steepest descent at the saddle-point $u_{0,+}$. Let $v\in\mathbb{C}$ point in that direction. Then, by definition, $v^2\phi''(u_{0,+})$ is real and positive, where ϕ is as in 5.6. Thus $\arg(v) = -\arg(\phi''(u_{0,+}))/2$. By (5.9),

$$\arg(\phi''(u_{0,+})) = \arg(-i\ell u_{0,+} + 2i\tau) - 2\arg(u_{0,+}).$$

Starting as in (5.18), we obtain that

$$\arg(-i\ell u_{0,+} + 2i\tau) = \arctan\left(\frac{\rho - \sqrt{\frac{j-1}{2}}}{-1 + \sqrt{\frac{j+1}{2}}}\right),$$

and

(5.20)

$$\begin{aligned}\frac{\rho - \sqrt{\frac{j-1}{2}}}{-1 + \sqrt{\frac{j+1}{2}}} &= \frac{\left(\rho - \sqrt{\frac{j-1}{2}}\right)\left(1 + \sqrt{\frac{j+1}{2}}\right)}{-1 + \frac{j+1}{2}} = \frac{\rho - \sqrt{2(j-1)} + \rho\sqrt{2(j+1)}}{j-1} \\ &= \frac{\rho + \sqrt{\frac{2}{j+1}}\left(-\sqrt{j^2-1} + \rho\cdot(j+1)\right)}{j-1} = \frac{\rho + \frac{1}{\upsilon}(-\rho + \rho\cdot(j+1))}{j-1} \\ &= \frac{\rho(1 + j/\upsilon)}{j-1} = \frac{(j+1)(1+j/\upsilon)}{\rho} = \frac{2\upsilon(\upsilon+j)}{\rho}.\end{aligned}$$

Hence, by (5.13),

$$\arg(\phi''(u_{0,+})) = \arctan\frac{2\upsilon(\upsilon+j)}{\rho} - \arccos\upsilon(\rho)^{-1}.$$

Therefore, the direction of steepest descent is

$$\begin{aligned}\arg(v) &= -\frac{\arg(\phi''(u_{0,+}))}{2} = \arg(u_{0,+}) - \frac{1}{2}\arctan\frac{2\upsilon(\upsilon+j)}{\rho} \\ &= \arg(u_{0,+}) - \arctan\Upsilon,\end{aligned} \tag{5.21}$$

where

$$\Upsilon = \tan\frac{1}{2}\arctan\frac{2\upsilon(\upsilon+j)}{\rho}. \tag{5.22}$$

Since

$$\tan\frac{\alpha}{2} = \frac{1}{\sin\alpha} - \frac{1}{\tan\alpha} = \sqrt{1 + \frac{1}{\tan^2\alpha}} - \frac{1}{\tan\alpha},$$

we see that

$$\Upsilon = \left(\sqrt{1 + \frac{\rho^2}{4\upsilon^2(\upsilon+j)^2}} - \frac{\rho}{2\upsilon(\upsilon+j)}\right). \tag{5.23}$$

Recall as well that

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}, \qquad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}.$$

Hence, if we let

$$\theta_0 = \arg(u_{0,+}) = \frac{1}{2}\arccos\frac{1}{\upsilon(\rho)}, \tag{5.24}$$

we get that

$$\tag{5.25} \begin{aligned} \cos\theta_0 &= \cos\left(\frac{1}{2}\arccos\frac{1}{\upsilon(\rho)}\right) = \sqrt{\frac{1}{2}+\frac{1}{2\upsilon(\rho)}},\\ \sin\theta_0 &= \sin\left(\frac{1}{2}\arccos\frac{1}{\upsilon(\rho)}\right) = \sqrt{\frac{1}{2}-\frac{1}{2\upsilon(\rho)}}. \end{aligned}$$

We will prove now the useful inequality

$$\tag{5.26} \arctan\Upsilon > \theta_0,$$

i.e., $\arg(v) < 0$. By (5.21), (5.22) and (5.24), this is equivalent to $\arccos(1/\upsilon) \leq \arctan 2\upsilon(\upsilon+j)/\rho$. Since $\tan\alpha = \sqrt{1/\cos^2\alpha - 1}$, we know that $\arccos(1/\upsilon) = \arctan\sqrt{\upsilon^2-1}$; thus, in order to prove (5.26), it is enough to check that

$$\sqrt{\upsilon^2-1} \leq \frac{2\upsilon(\upsilon+j)}{\rho}.$$

This is easy, since $j > \rho$ and $\sqrt{\upsilon^2-1} < \upsilon < 2\upsilon$.

5.3.4. *The contour.* We must now choose the contour of integration. First, let us discuss our options. By (C.9), $\Upsilon \geq 0.79837$; moreover, it is easy to show that Υ tends to 1 when either $\rho \to 0^+$ or $\rho \to \infty$. This means that neither the simplest contour (a straight ray from the origin within the first quadrant) nor what is arguably the second simplest contour (leaving the origin on a ray within the first quadrant, then sliding down a circle centered at the origin, passing past the saddle point until you reach the x-axis, which you then follow to infinity) are acceptable: either contour passes through the saddle point on a direction close to 45 degrees (= arctan(1)) off from the direction of steepest descent. (The saddle-point method allows in principle for any direction less than 45 degrees off from the direction of steepest descent, but the bounds can degrade rapidly – by more than a constant factor – when 45 degrees are approached.)

It is thus best to use a curve that goes through the saddle point $u_{+,0}$ in the direction of steepest descent. We thus should use an element of a two-parameter family of curves. The curve should also have a simple description in terms of polar coordinates.

We decide that our contour C will be a *limaçon of Pascal*. (Excentric circles would have been another reasonable choice.) Let C be parameterized by

$$\tag{5.27} y = \left(-\frac{c_1}{\ell}r + c_0\right)r, \qquad x = \sqrt{r^2-y^2}$$

for $r \in [(c_0-1)\ell/c_1, c_0\ell/c_1]$, where c_0 and c_1 are parameters to be set later. The curve goes from $(0,(c_0-1)\ell/c_1)$ to $(c_0\ell/c_1, 0)$, and stays within the first quadrant.[7] In order for the curve to go through the point $u_{0,+}$, we must have

$$\tag{5.28} -\frac{c_1 r_0}{\ell} + c_0 = \sin\theta_0,$$

where

$$\tag{5.29} r_0 = |u_{0,+}| = \frac{\ell}{\sqrt{2}}\sqrt{\upsilon(\rho)^2-\upsilon(\rho)},$$

[7] Because $c_0 \geq 1$, by (C.21).

and θ_0 and $\sin\theta_0$ are as in (5.24) and (5.25). We must also check that the curve C goes through $u_{0,+}$ in the direction of steepest descent. The argument of the point (x,y) is

$$\theta = \arcsin\frac{y}{r} = \arcsin\left(-\frac{c_1 r}{\ell} + c_0\right).$$

Hence

$$r\frac{d\theta}{dr} = r\frac{d\arcsin\left(-\frac{c_1 r}{\ell} + c_0\right)}{dr} = r\cdot\frac{-c_1/\ell}{\cos\arcsin\left(-\frac{c_1 r}{\ell} + c_0\right)} = \frac{-c_1 r}{\ell\cos\theta}.$$

This means that, if v is tangent to C at the point $u_{0,+}$,

$$\tan(\arg(v) - \arg(u_{0,+})) = r\frac{d\theta}{dr} = \frac{-c_1 r_0}{\ell\cos\theta_0},$$

and so, by (5.21),

$$c_1 = \frac{\ell\cos\theta_0}{r_0}\Upsilon, \tag{5.30}$$

where Υ is as in (5.22). In consequence,

$$c_0 = \frac{c_1 r_0}{\ell} + \sin\theta_0 = (\cos\theta_0)\cdot\Upsilon + \sin\theta_0,$$

and so, by (5.25),

$$c_1 = \sqrt{\frac{1+1/\upsilon}{\upsilon^2-\upsilon}}\cdot\Upsilon, \qquad c_0 = \sqrt{\frac{1}{2}+\frac{1}{2\upsilon}}\cdot\Upsilon + \sqrt{\frac{1}{2}-\frac{1}{2\upsilon}}. \tag{5.31}$$

Incidentally, we have also shown that the arc-length infinitesimal is

$$|du| = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}dr = \sqrt{1+\frac{(c_1 r/\ell)^2}{\cos^2\theta}}dr = \sqrt{1+\frac{r^2}{\frac{\ell^2}{c_1^2}-\left(\frac{c_0}{c_1}\ell - r\right)^2}}dr. \tag{5.32}$$

The contour will be as follows: first we go out of the origin along a straight radial segment C_1; then we meet the curve C, and we follow it clockwise for a segment C_2, with the saddle-point roughly at its midpoint; then we follow another radial ray C_3 up to infinity. For ρ small, C_3 will just be the x-axis. Both C_1 and C_3 will be contained within the first quadrant; we will specify them later.

5.3.5. *The integral over the main contour segment C_2.* We recall that

$$\phi(u) = \frac{u^2}{2} + \ell i u - i\tau\log u. \tag{5.33}$$

Our aim is now to bound the integral

$$\int_{C_2} e^{-\Re(\phi(u))}u^{\sigma-1}du$$

over the main contour segment C_2. We will proceed as follows. First, we will parameterize the integral using a real variable ν, with the value $\nu = 0$ corresponding to the saddle point $u = u_{0,+}$. We will bound $\Re(\phi(u))$ from below by an expression of the form $\Re(\phi(u_{0,+})) + \eta\nu^2$. We then bound $|u|^{\sigma-1}|du/d\nu|$ from above by a constant. This procedure will give a bound that is larger than the true value by at most a (very moderate) constant factor.

For $u = x + iy$ (or (r,θ) in polar coordinates), (5.33) gives us

$$\Re(\phi(u)) = \frac{x^2 - y^2}{2} - \ell y + \theta\tau = \frac{r^2 - 2y^2}{2} - \ell y + \tau \arcsin\frac{y}{r} \tag{5.34}$$
$$= \left(\frac{4\tau}{\ell}\right)^2 \psi_0(\nu) = \ell^2\rho^2\psi_0(\nu),$$

where, by (5.27), (5.28), and (5.31),

$$\psi_0(\nu) = \frac{(\nu+\nu_0)^2}{2}(1 - 2(\sin\theta_0 - c_1\rho\nu)^2)$$
$$- \frac{\nu+\nu_0}{\rho}(\sin\theta_0 - c_1\rho\nu) + \frac{\arcsin(\sin\theta_0 - c_1\rho\nu)}{4\rho},$$

and

$$\nu = \frac{r - r_0}{\ell\rho}, \qquad \nu_0 = \frac{r_0}{\ell\rho}. \tag{5.35}$$

By (5.27), (5.28) and (5.35),

$$\frac{y}{r} = c_0 - c_1\rho(\nu+\nu_0) = \sin\theta_0 - c_1\rho\nu \tag{5.36}$$

and so

$$c_0 - c_1\nu_0\rho = \sin\theta_0. \tag{5.37}$$

The variable ν will range within an interval

$$[\alpha_0, \alpha_1] \subset \left(-\frac{1-\sin\theta_0}{c_1\rho}, \frac{\sin\theta_0}{c_1\rho}\right]. \tag{5.38}$$

(Here $\nu = -(1-\sin\theta_0)/(c_1\rho)$ corresponds to the intersection with the y-axis, and $\nu = (\sin\theta_0)/(c_1\rho)$ corresponds to the intersection with the x-axis.)

We work out the expansions around 0 of

(5.39)
$$\frac{(\nu+\nu_0)^2}{2}(1 - 2(\sin\theta_0 - c_1\rho\nu)^2) = \frac{\nu_0^2\cos 2\theta_0}{2} + (\nu_0\cos 2\theta_0 + 2\nu_0^2 c_1\rho\sin\theta_0)\nu$$
$$+ \left(\frac{\cos 2\theta_0}{2} + 4c_1\nu_0\rho\sin\theta_0 - c_1^2\rho^2\nu_0^2\right)\nu^2$$
$$+ 2(-\nu_0 c_1^2\rho^2 + c_1\rho\sin\theta_0)\nu^3 - c_1^2\rho^2\nu^4,$$
$$-\frac{\nu+\nu_0}{\rho}(\sin\theta_0 - c_1\rho\nu) = -\frac{\nu_0\sin\theta_0}{\rho} + \left(-\frac{\sin\theta_0}{\rho} + c_1\nu_0\right)\nu + c_1\nu^2,$$
$$\frac{\arcsin(\sin\theta_0 - c_1\rho\nu)}{4\rho} = \frac{\theta_0}{4\rho} + \frac{1}{4\rho}\sum_{k=1}^{\infty}\frac{P_k(\sin\theta_0)}{(\cos\theta_0)^{2k-1}}\frac{(-c_1\rho)^k}{k!}\nu^k$$
$$= \frac{\theta_0}{4\rho} + \frac{1}{4\rho}\left(\frac{-c_1\rho}{\cos\theta_0}\nu + \frac{(c_1\rho)^2\sin\theta_0}{2(\cos\theta_0)^3}\nu^2 + \dots\right),$$

where $P_1(t) = 1$ and $P_{k+1}(t) = P_k'(t)(1-t^2) + (2k-1)tP(t)$ for $k \geq 1$. (This follows from $(\arcsin z)' = 1/\sqrt{1-z^2}$; it is easy to show that $(\arcsin z)^{(k)} = P_k(z)(1 - z^2)^{-(k-1/2)}$.)

We sum these three expressions and obtain a series $\psi_0(\nu) = \sum_k a_k\nu^k$. We already know that

(1) a_0 equals the value of $\Re(\phi(u))/(\ell^2\rho^2)$ at the saddle point $u_{0,+}$,
(2) $a_1 = 0$,

(3)
$$a_2 = \frac{1}{2}\left(\frac{1}{\ell\rho}\right)^2 \left(\frac{dr}{d\nu}\right)^2 |\phi''(u_{0,+})| \left|\frac{du}{dr}|_{r=r_0}\right|^2 = \frac{1}{2}|\phi''(u_{0,+})| \left|\frac{du}{dr}|_{r=r_0}\right|^2.$$
Here, as we know from (5.9), (5.19) and (5.17),
$$|\phi''(u_{0,+})| = \frac{|-i\ell u_{0,+} + 2i\tau|}{|u_{0,+}|^2} = \frac{\ell^2\sqrt{\frac{j(\rho)}{2}}\sqrt{\upsilon^2-\upsilon}}{\frac{\ell^2}{2}(\upsilon^2-\upsilon)} = \sqrt{\frac{2j(\rho)}{\upsilon^2-\upsilon}},$$
and, by (5.31) and (5.32),

(5.40)
$$\begin{aligned}\left|\frac{du}{dr}|_{r=r_0}\right| = \frac{|du|}{|dr|}|_{r=r_0} &= \sqrt{1+\frac{r_0^2}{\frac{\ell^2}{c_1^2} - \left(\frac{c_0}{c_1}\ell - r_0\right)^2}} = \sqrt{1+\frac{c_1^2}{\ell^2}\frac{r_0^2}{1-\sin^2\theta_0}} \\ &= \sqrt{1+\frac{c_1^2}{\ell^2}\frac{\frac{\ell^2}{2}(\upsilon^2-\upsilon)}{\frac{1}{2}+\frac{1}{2\upsilon}}} = \sqrt{1+c_1^2\frac{\upsilon^2-\upsilon}{1+1/\upsilon}} \\ &= \sqrt{1+\Upsilon^2}.\end{aligned}$$

Thus,
$$a_2 = \frac{1}{2}\sqrt{\frac{2j(\rho)}{\upsilon^2-\upsilon}}(1+\Upsilon^2),$$
where Υ is as in (5.23).

Let us simplify our expression for $\psi_0(\nu)$ somewhat. We can replace the third series in (5.39) by a truncated Taylor series ending at $k=2$, namely,
$$\frac{\arcsin(\sin\theta_0 - c_1\rho\nu)}{4\rho} = \frac{\theta_0}{4\rho} + \frac{1}{4\rho}\left(\frac{-c_1\rho}{\cos\theta_0}\nu + \frac{(c_1\rho)^2\sin\theta_1}{2(\cos\theta_1)^3}\nu^2\right)$$
for some θ_1 between θ_0 and θ. Then $\theta_1 \in [0,\pi/2]$, and so
$$\frac{\arcsin(\sin\theta_0 - c_1\rho\nu)}{4\rho} \geq \frac{\theta_0}{4\rho} + \frac{1}{4\rho}\cdot\frac{-c_1\rho}{\cos\theta_0}\nu.$$

Since
$$R(\nu) = -c_1^2\rho^2\nu^2 + 2(\sin\theta_0 - c_1\rho\nu_0)c_1\rho\nu$$
is a quadratic with negative leading coefficient, its minimum within $[-\alpha_0,\alpha_1]$ (see (5.38)) is bounded from below by $\min(R(-(1-\sin\theta_0)/(c_1\rho)), R((\sin\theta_0)/(c_1\rho)))$. We compare
$$R\left(\frac{\sin\theta_0}{c_1\rho}\right) = 2c_3\sin\theta_0 - \sin^2\theta_0,$$
where $c_3 = \sin\theta_0 - c_1\rho\nu_0$, and
$$\begin{aligned}R\left(-\frac{1-\sin\theta_0}{c_1\rho}\right) &= -2c_3(1-\sin\theta_0) - (1-\sin\theta_0)^2 \\ &= 2c_3\sin\theta_0 - \sin^2\theta_0 - 2c_3 - 1 + 2\sin\theta_0\end{aligned}.$$
The question is whether
$$\begin{aligned}R\left(-\frac{1-\sin\theta_0}{c_1\rho}\right) - R\left(\frac{\sin\theta_0}{c_1\rho}\right) &= -2c_3 - 1 + 2\sin\theta_0 \\ &= -2(\sin\theta_0 - c_1\rho\nu_0) - 1 + 2\sin\theta_0 \\ &= 2c_1\rho\nu_0 - 1\end{aligned}$$

is positive. It is:

$$c_1\rho\nu_0 = \frac{c_1\rho_0}{\ell} = c_1\sqrt{\frac{\upsilon^2-\upsilon}{2}} = \sqrt{\frac{1+1/\upsilon}{2}}\cdot\Upsilon \geq \frac{\Upsilon}{\sqrt{2}},$$

and, as we know from (C.9), $\Upsilon > 0.79837$ is greater than $1/\sqrt{2} = 0.70710\ldots$. Hence, by (5.37),

$$\begin{aligned} R(\nu) \geq R\left(\frac{\sin\theta_0}{c_1\rho}\right) &= 2c_3\sin\theta_0 - \sin^2\theta_0 = \sin^2\theta_0 - 2c_1\rho\nu_0\sin\theta_0 \\ &= \sin^2\theta_0 - 2(c_0-\sin\theta_0)\sin\theta_0 = 3\sin^2\theta_0 - 2c_0\sin\theta_0 \\ &= 3\sin^2\theta_0 - 2((\cos\theta_0)\cdot\Upsilon + \sin\theta_0)\sin\theta_0 = \sin^2\theta_0 - (\sin 2\theta_0)\cdot\Upsilon. \end{aligned}$$

We conclude that

$$\psi_0(\nu) \geq \frac{\Re(\phi(u_{0,+}))}{\ell^2\rho^2} + \eta\nu^2, \tag{5.41}$$

where

$$\eta = a_2 - \frac{1}{4\rho}\frac{(c_1\rho)^2\sin\theta_0}{2(\cos\theta_0)^3} + \sin^2\theta_0 - (\sin 2\theta_0)\cdot\Upsilon. \tag{5.42}$$

We can simplify this further, using

$$\begin{aligned} \frac{1}{4\rho}\frac{(c_1\rho)^2\sin\theta_0}{2(\cos\theta_0)^3} &= \frac{\rho}{8}\cdot\frac{1+1/\upsilon}{\upsilon^2-\upsilon}\cdot\Upsilon^2\cdot\frac{\sqrt{\frac{1}{2}-\frac{1}{2\upsilon}}}{\left(\frac{1}{2}+\frac{1}{2\upsilon}\right)^{3/2}} = \frac{\rho}{4}\frac{\Upsilon^2}{\upsilon^2-\upsilon}\frac{\sqrt{1-1/\upsilon}}{\sqrt{1+1/\upsilon}} \\ &= \frac{\rho}{4}\frac{\Upsilon^2}{\upsilon\sqrt{\upsilon^2-1}} = \frac{\rho}{4}\frac{\Upsilon^2}{\sqrt{\frac{j+1}{2}}\sqrt{\frac{j-1}{2}}} = \frac{\rho}{4}\frac{\Upsilon^2}{\sqrt{\rho^2/4}} = \frac{\Upsilon^2}{2} \end{aligned}$$

and (by (5.25))

$$\begin{aligned} \sin 2\theta_0 &= 2\sin\theta_0\cos\theta_0 = 2\sqrt{\frac{1}{4}-\frac{1}{4\upsilon^2}} = \frac{\sqrt{\upsilon^2-1}}{\upsilon} \\ &= \frac{\upsilon\sqrt{\upsilon^2-1}}{\upsilon^2} = \frac{\rho/2}{(j+1)/2} = \frac{\rho}{j+1}. \end{aligned}$$

Therefore (again by (5.25))

$$\eta = \frac{1}{2}\sqrt{\frac{2j}{\upsilon^2-\upsilon}}(1+\Upsilon^2) - \frac{\Upsilon^2}{2} + \frac{1}{2} - \frac{1}{2\upsilon} - \frac{\rho}{j+1}\cdot\Upsilon. \tag{5.43}$$

Now recall that our task is to bound the integral

$$\begin{aligned} \int_{C_2} e^{-\Re(\phi(u))}|u|^{\sigma-1}|du| &= \int_{\alpha_0}^{\alpha_1} e^{-\ell^2\rho^2\psi_0(\nu)}(\ell\rho(\nu+\nu_0))^{\sigma-1}\left|\frac{du}{dr}\cdot\frac{dr}{d\nu}\right| d\nu \\ &\leq (\ell\rho)^\sigma e^{-\Re(\phi(u_{0,+}))}\cdot\int_{\alpha_0}^{\alpha_1} e^{-\eta\ell^2\rho^2\cdot\nu^2}(\nu+\nu_0)^{\sigma-1}\left|\frac{du}{dr}\right| d\nu. \end{aligned} \tag{5.44}$$

(We are using (5.34) and (5.41).) Since $u_{0,+}$ is a solution to equation (5.7), we see from (5.6) that

$$\begin{aligned} \Re(\phi(u_{0,+})) &= \Re\left(\frac{u_{0,+}^2}{2} + \ell i u_{0,+} - i\tau\log u_{0,+}\right) \\ &= \Re\left(\frac{\ell i u_{0,+}}{2} + \frac{i\tau}{2} + \tau\arg(u_{0,+})\right) = \frac{1}{2}\Re(\ell i u_{0,+}) + \tau\arg(u_0,+). \end{aligned}$$

We defined $y = \Re(-\ell i u_0)$ (after (5.10)), and we computed $y/2 - \arg(u_{0,+})\tau$ in (5.14). This gives us

$$e^{-\Re(\phi(u_{0,+}))} = e^{-\left(\arccos\frac{1}{\upsilon} - \frac{2(\upsilon-1)}{\rho}\right)\frac{\tau}{2}}.$$

If $\sigma \leq 1$, we can bound

$$(\nu+\nu_0)^{\sigma-1} \leq \begin{cases} \nu_0^{\sigma-1} & \text{if } \nu \geq 0,\\ (\alpha_0+\nu_0)^{\sigma-1} & \text{if } \nu < 0,\end{cases} \tag{5.45}$$

provided that $\alpha_0 + \nu_0 > 0$ (as will be the case). If $\sigma > 1$, then

$$(\nu+\nu_0)^{\sigma-1} \leq \begin{cases} \nu_0^{\sigma-1} & \text{if } \nu \leq 0,\\ (\alpha_1+\nu_0)^{\sigma-1} & \text{if } \nu > 0.\end{cases}$$

By (5.32),

$$\left|\frac{du}{dr}\right| = \sqrt{1 + \frac{(c_1 r/\ell)^2}{\cos^2\theta}} = \sqrt{1 + \frac{(c_1\rho(\nu+\nu_0))^2}{1-(\sin\theta_0 - c_1\rho\nu)^2}}$$

(This diverges as $\theta \to \pi/2$; this is main reason why we cannot actually follow the curve all the way to the y-axis.) Since we are aiming at a bound that is tight only up to an order of magnitude, we can be quite brutal here, as we were when using (5.45): we bound $(c_1 r/\ell)^2$ from above by its value when the curve meets the x-axis (i.e., when $r = c_0\ell/c_1$). We bound $\cos^2\theta$ from below by its value when $\nu = \alpha_1$. We obtain

$$\left|\frac{du}{dr}\right| = \sqrt{1 + \frac{c_0^2}{1-(\sin\theta_0 - c_1\rho\alpha_1)^2}} = \sqrt{1 + \frac{c_0^2}{\cos^2\theta_-}},$$

where θ_- is the value of θ when $\nu = \alpha_1$.

Finally, we complete the integral in (5.44), we split it in two (depending on whether $\nu \geq 0$ or $\nu < 0$) and use

$$\int_0^\alpha e^{\eta\ell^2\rho^2\cdot\nu^2} d\nu \leq \frac{1}{\ell\rho\sqrt{\eta}}\int_0^\infty e^{-\nu^2} d\nu = \frac{\sqrt{\pi}/2}{\ell\rho\sqrt{\eta}}.$$

Therefore,

$$\begin{aligned} &\int_{C_2} e^{-\Re(\phi(u))}|u|^{\sigma-1}|du|\\ &= (\ell\rho)^\sigma e^{-\left(\arccos\frac{1}{\upsilon} - \frac{2(\upsilon-1)}{\rho}\right)\frac{\tau}{2}} \sqrt{1+\frac{c_0^2}{\cos^2\theta_-}}\cdot\frac{\sqrt{\pi}/2}{\ell\rho\sqrt{\eta}}\cdot\left(\nu_0^{\sigma-1} + (\alpha_{j_\sigma}+\nu_0)^{\sigma-1}\right)\\ &= \frac{\sqrt{\pi}}{2} r_0^{\sigma-1}\left(\left(1+\left(1+\frac{\alpha_{j_\sigma}}{\nu_0}\right)^{\sigma-1}\right)\sqrt{1+\frac{c_0^2}{\cos^2\theta_-}}\right)\frac{e^{-\left(\arccos\frac{1}{\upsilon}-\frac{2(\upsilon-1)}{\rho}\right)\frac{\tau}{2}}}{\sqrt{\eta}}, \end{aligned} \tag{5.46}$$

where $j_\sigma = 0$ if $\sigma \leq 1$ and $j_\sigma = 1$ if $\sigma > 1$. We can set $\alpha_1 = (\sin\theta_0)/(c_1\rho)$. We can also express $\alpha_0 + \nu_0$ in terms of θ_-:

$$\alpha_0 + \nu_0 = \frac{r_-}{\ell\rho} = \frac{(c_0 - \sin\theta_-)\frac{\ell}{c_1}}{\ell\rho} = \frac{c_0 - \sin\theta_-}{c_1\rho}. \tag{5.47}$$

Since $\nu_0 = r_0/(\ell\rho)$ (by (5.35)) and r_0 is as in (5.29),

$$\nu_0 = \frac{\sqrt{\upsilon(\rho)^2 - \upsilon(\rho)}}{\sqrt{2}\rho}.$$

Definition (5.22) implies immediately that $\Upsilon \leq 1$. Thus, by (5.31),

$$c_1\rho\nu_0 = \Upsilon \cdot \sqrt{2(1+1/\upsilon)} \leq 2\Upsilon \leq 2, \tag{5.48}$$

while, by (C.9),

$$c_1\rho\nu_0 = \Upsilon \cdot \sqrt{2(1+1/\upsilon)} \geq 0.79837 \cdot \sqrt{2} \tag{5.49}$$

By (5.47) and (5.48),

$$\left(1 + \frac{\alpha_0}{\nu_0}\right)^{-1} = \frac{\nu_0}{\alpha_0 + \nu_0} = \frac{c_1\rho\nu_0}{c_0 - \sin\theta_-} \leq \frac{2}{c_0 - \sin\theta_-}, \tag{5.50}$$

whereas

$$\left(1 + \frac{\alpha_1}{\nu_0}\right) = 1 + \frac{\sin\theta_0}{c_1\rho\nu_0} \leq 1 + \frac{1/\sqrt{2}}{0.79837 \cdot \sqrt{2}} \leq 1.62628.$$

We will now use some (rigorous) numerical bounds, proven in Appendix C.2. First of all, by (C.21), $c_0 > 1$ for all $\rho > 0$; this assures us that $c_0 - \sin\theta_- > 0$, and so the last expression in (5.50) is well defined. By (5.47), this also shows that $\alpha_0 + \nu_0 > 0$, i.e., the curve C stays within the first quadrant for $0 \leq \theta \leq \pi/2$, as we said before.

We would also like to have an upper bound for

$$\sqrt{\frac{1}{\eta}\left(1 + \frac{c_0^2}{\cos^2\theta_-}\right)}, \tag{5.51}$$

using (5.43). With this in mind, we finally choose θ_-:

$$\theta_- = \frac{\pi}{4}. \tag{5.52}$$

Thus, by (C.36),

$$\sqrt{\frac{1}{\eta}\left(1 + \frac{c_0^2}{\cos^2\theta_-}\right)} \leq \sqrt{\frac{1+2c_0^2}{\eta}} \leq \sqrt{\min(5, 0.86\rho)} \leq \min(\sqrt{5}, 0.93\sqrt{\rho}).$$

We also get

$$\frac{2}{c_0 - \sin\theta_-} \leq \frac{2}{1 - 1/\sqrt{2}} \leq 7.82843.$$

Finally, by (C.39),

$$\sqrt{\frac{\upsilon^2 - \upsilon}{2}} \geq \begin{cases} \rho/6 & \text{if } \rho \leq 4, \\ \frac{\sqrt{\rho}}{2} - \frac{1}{2^{3/2}} \leq \left(1 - \frac{1}{2^{3/2}}\right)\frac{\sqrt{\rho}}{2} & \text{if } \rho > 4 \end{cases}$$

and so, since $\rho\ell = 4\tau/\ell$, $\sqrt{\rho}\ell = 2\sqrt{\tau}$ and $(1 - 1/2^{3/2}) \leq 2/3$, (5.29) gives us

$$r_0 \geq \begin{cases} \frac{2}{3}\frac{\tau}{\ell} & \text{if } \tau \leq \ell^2 \\ \frac{2}{3}\sqrt{\tau} & \text{if } \tau > \ell^2 \end{cases} = \frac{2}{3}\min\left(\frac{\tau}{\ell}, \sqrt{\tau}\right).$$

We conclude that

$$\int_{C_2} e^{-\Re(\phi(u))}|u|^{\sigma-1}|du| = C_{\tau,\ell} \cdot e^{-\left(\arccos\frac{1}{\upsilon} - \frac{2(\upsilon-1)}{\rho}\right)\frac{\tau}{2}}, \tag{5.53}$$

where
(5.54)
$$C_{\tau,\ell} = \min\left(2, \frac{3.3\sqrt{\tau}}{\ell}\right)\left(1+\max\left(7.83^{1-\sigma}, 1.63^{\sigma-1}\right)\right)\left(\frac{3/2}{\min(\tau/\ell, \sqrt{\tau})}\right)^{1-\sigma}$$

for all $\tau > 0$, $\ell > 0$ and all σ. By reflection on the x-axis, the same bound holds for $\tau < 0$, $\ell < 0$ and all σ. Lastly, (5.53) is also valid for $\ell = 0$, provided we replace (5.54) and the exponent of (5.53) by their limits as $\ell \to 0^+$.

5.3.6. *The integral over the rest of the contour.* It remains to complete the contour. Since we have set $\theta_- = \pi/4$, C_1 will be a segment of the ray at 45 degrees from the x-axis, counterclockwise (i.e., $y = x$, $x \geq 0$). The segment will go from $(0,0)$ up to $(x,y) = (r_-/\sqrt{2}, r_-/\sqrt{2})$, where, by (5.27),
$$\frac{1}{\sqrt{2}} = \frac{y}{r_-} = -\frac{c_1}{\ell}r + c_0,$$
and so
$$r_- = \frac{\ell}{c_1}\left(c_0 - \frac{1}{\sqrt{2}}\right). \tag{5.55}$$
Let $w = (1+i)/\sqrt{2}$. Looking at (5.6), we see that
(5.56)
$$\begin{aligned}\left|\int_{C_1} e^{-u^2/2} e(\delta u) u^{s-1} du\right| &= \left|\int_{C_1} e^{-\phi(u)} u^{\sigma-1} du\right| \\ &\leq \int_{C_1} e^{-\Re(\phi(u))} |u|^{\sigma-1} |du| = \int_0^{r_-} e^{-\Re(\phi(tw))} t^{\sigma-1} dt,\end{aligned}$$
where $\phi(u)$ is as in (5.33). Here
$$\Re(\phi(tw)) = \Re\left(\frac{t^2}{2}i + \ell i w t - i\tau\left(\log t + i\frac{\pi}{4}\right)\right) = -\frac{\ell t}{\sqrt{2}} + \frac{\pi}{4}\tau, \tag{5.57}$$
and, by (C.40),
$$-\frac{\ell r_-}{\sqrt{2}} + \frac{\pi}{4}\tau \geq -0.07639\ell^2\rho + \frac{\pi}{4}\tau = \left(\frac{\pi}{4} - 0.30556\right)\tau > 0.4798\tau.$$
Consider first the case $\sigma \geq 1$. Then
$$\int_0^{r_-} e^{-\Re(\phi(tw))} t^{\sigma-1} dt \leq r_-^{\sigma-1} \int_0^{r_-} e^{\frac{\ell t}{\sqrt{2}} - \frac{\pi}{4}\tau} dt \leq r_-^{\sigma} e^{\frac{\ell r_-}{\sqrt{2}} - \frac{\pi}{4}\tau}.$$
By (5.55) and (C.40),
$$r_- \leq \sqrt{\rho}\ell/2 \leq \sqrt{\tau}, \tag{5.58}$$
Hence, for $\sigma \geq 1$,
$$\left|\int_{C_1} e^{-u^2/2} e(\delta u) u^{s-1} du\right| \leq \tau^{\sigma/2} e^{-0.4798\tau}. \tag{5.59}$$

Assume now that $0 \leq \sigma < 1$, $s \neq 0$. We can see that it is wise to start by an integration by parts, so as to avoid convergence problems arising from the term $t^{\sigma-1}$ within the integral as $\sigma \to 0^+$. We have
$$\int_{C_1} e^{-u^2/2} e(\delta u) u^{s-1} du = e^{-u^2/2} e(\delta u) \frac{u^s}{s}\Big|_0^{wr_-} - \int_{C_1} \left(e^{-u^2/2} e(\delta u)\right)' \frac{u^s}{s} du.$$

By (5.57),

$$\left| e^{-u^2/2} e(\delta u) \frac{u^s}{s} |_0^{wr_-} \right| = e^{-\Re(\phi(wr_-))} \cdot \frac{r_-^\sigma}{|s|} \leq \frac{r_-^\sigma}{\tau} \cdot e^{\frac{\ell r_-}{\sqrt{2}} - \frac{\pi}{4}\tau}$$

As for the integral,

(5.60)
$$\begin{aligned}\int_{C_1} \left(e^{-u^2/2} e(\delta u)\right)' \frac{u^s}{s} du &= -\int_{C_1} (u + \ell i) e^{-u^2/2 - \ell i u} \frac{u^s}{s} du \\ &= -\frac{1}{s} \int_{C_1} e^{-u^2/2} e(\delta u) u^{s+1} du - \frac{\ell i}{s} \int_{C_1} e^{-u^2/2} e(\delta u) u^s du.\end{aligned}$$

Hence, by (5.56) and (5.57),

$$\begin{aligned}\left| \int_{C_1} \left(e^{-u^2/2} e(\delta u)\right)' \frac{u^s}{s} du \right| &\leq \frac{1}{|s|} \int_0^{r_-} e^{\frac{\ell t}{\sqrt{2}} - \frac{\pi}{4}\tau} t^{\sigma+1} dt + \frac{\ell}{|s|} \int_0^{r_-} e^{\frac{\ell t}{\sqrt{2}} - \frac{\pi}{4}\tau} t^\sigma dt \\ &\leq \left(\frac{r_-^{\sigma+1}}{|s|} + \frac{\ell r_-^\sigma}{|s|} \right) \int_0^{r_-} e^{\frac{\ell t}{\sqrt{2}} - \frac{\pi}{4}\tau} dt \\ &\leq \frac{r_-^{\sigma+1} + \ell r_-^\sigma}{\tau} \cdot \min\left(\frac{\sqrt{2}}{\ell}, r_- \right) \cdot e^{\frac{\ell r_-}{2} - \frac{\pi}{4}\tau} \\ &\leq \left(\frac{r_-^{\sigma+2}}{\tau} + \frac{\sqrt{2} r_-^\sigma}{\tau} \right) \cdot e^{\frac{\ell r_-}{2} - \frac{\pi}{4}\tau}.\end{aligned}$$

By (5.58),

$$\left(\frac{r_-^{\sigma+2}}{\tau} + \frac{(1+\sqrt{2}) r_-^\sigma}{\tau} \right) \leq \frac{\tau^{\frac{\sigma}{2}+1} + (1+\sqrt{2}) \tau^{\frac{\sigma}{2}}}{\tau}.$$

We conclude that

$$\begin{aligned}\left| \int_{C_1} e^{-u^2/2} e(\delta u) u^{s-1} du \right| &\leq \frac{\tau^{\frac{\sigma}{2}+1} + (1+\sqrt{2}) \tau^{\frac{\sigma}{2}}}{\tau} \cdot e^{-\frac{\ell r_-}{2} + \frac{\pi}{4}\tau} \\ &\leq \left(1 + \frac{1+\sqrt{2}}{\tau}\right) \tau^{\frac{\sigma}{2}} \cdot e^{-0.4798\tau}\end{aligned}$$

when $\sigma \in [0, 1)$; by (5.59), this is true for $\sigma \geq 1$ as well.

Now let us examine the contribution of the last segment C_3 of the contour. Since C_2 hits the x-axis at $c_0\ell/c_1$, we define C_3 to be the segment of the x-axis going from $x = c_0\ell/c_1$ till $x = \infty$. Then

(5.61)
$$\left| \int_{C_3} e^{-t^2/2} e(\delta t) t^s \frac{dt}{t} \right| = \left| \int_{\frac{c_0\ell}{c_1}}^\infty e^{-x^2/2} e(\delta x) x^s \frac{dx}{x} \right| \leq \int_{\frac{c_0\ell}{c_1}}^\infty e^{-x^2/2} x^\sigma \frac{dx}{x}.$$

Now

$$\begin{aligned}\left(-e^{-x^2/2} x^{\sigma-2}\right)' &= e^{-x^2/2} x^{\sigma-1} - (\sigma-2) e^{-x^2/2} x^{\sigma-3} \\ \left(-e^{-x^2/2} (x^{\sigma-2} + (\sigma-2) x^{\sigma-4})\right)' &= e^{-x^2/2} x^{\sigma-1} - (\sigma-2)(\sigma-4) e^{-x^2/2} x^{\sigma-5}\end{aligned}$$

and so on, implying that

$$\begin{aligned}&\int_t^\infty e^{-x^2/2} x^\sigma \frac{dx}{x}\\ &\leq e^{-x^2/2}\cdot \begin{cases} x^{\sigma-2} & \text{if } 0\leq\sigma\leq 2,\\ \left(x^{\sigma-2}+(\sigma-2)x^{\sigma-4}\right) & \text{if } 2\leq\sigma\leq 4.\\ x^{\sigma-2}+(\sigma-2)x^{\sigma-4}+(\sigma-2)(\sigma-4)x^{\sigma-6} & \text{if } 4\leq\sigma\leq 6,\end{cases}\end{aligned}$$

and so on. By (C.43),

$$\frac{c_0\ell}{c_1} \geq \min\left(\frac{\tau}{\ell}, \frac{5}{4}\sqrt{\tau}\right).$$

We conclude that

$$\left|\int_{C_3} e^{-t^2/2} e(\delta t) t^s \frac{dt}{t}\right| \leq P_\sigma\left(\min\left(\frac{\tau}{\ell},\frac{5}{4}\sqrt{\tau}\right)\right) e^{-\min\left(\frac{1}{2}\left(\frac{\tau}{\ell}\right)^2, \frac{25}{32}\tau\right)},$$

where we can set $P_\sigma(x) = x^{\sigma-2}$ if $\sigma\in[0,2]$, $P_\sigma(x) = x^{\sigma-2}+(\sigma-2)x^{\sigma-4}$ if $\sigma\in[2,4]$ and $P_\sigma(x) = x^{\sigma-2}+(\sigma-2)x^{\sigma-4}+\ldots+(\sigma-2k)x^{\sigma-2(k+1)}$ if $\sigma\in[2k,2(k+1)]$.

* * *

We have left the case $\ell<0$ for the very end. In this case, we can afford to use a straight ray from the origin as our contour of integration. Let C' be the ray at angle $\pi/4-\alpha$ from the origin, i.e., $y = (\tan(\pi/4-\alpha))x$, $x>0$, where $\alpha>0$ is small. Write $v = e^{(\pi/4-\alpha)i}$. The integral to be estimated is

$$I = \int_{C'} e^{-u^2/2} e(\delta u) u^{s-1} du.$$

Let us try $\alpha=0$ first. Much as in (5.56) and (5.57), we obtain, for $\ell<0$,

$$\begin{aligned}|I| &\leq \int_0^\infty e^{-\left(\frac{-\ell t}{\sqrt{2}}+\frac{\pi}{4}\tau\right)} t^{\sigma-1}dt = e^{-\frac{\pi}{4}\tau}\int_0^\infty e^{-|\ell|t/\sqrt{2}} t^\sigma \frac{dt}{t}\\ &= e^{-\frac{\pi}{4}\tau}\cdot\left(\frac{\sqrt{2}}{|\ell|}\right)^\sigma \int_0^\infty e^{-t}t^\sigma\frac{dt}{t} = \left(\frac{\sqrt{2}}{|\ell|}\right)^\sigma \Gamma(\sigma)\cdot e^{-\frac{\pi}{4}\tau}\end{aligned} \tag{5.62}$$

for $\sigma>0$. Recall that $\Gamma(\sigma)\leq\sigma^{-1}$ for $0<\sigma<1$ (because $\sigma\Gamma(\sigma)=\Gamma(\sigma+1)$ and $\Gamma(\sigma)\leq 1$ for all $\sigma\in[1,2]$; the inequality $\Gamma(\sigma)\leq 1$ for $\sigma\in[1,2]$ can in turn be proven by $\Gamma(1)=\Gamma(2)=1$, $\Gamma'(1)<0<\Gamma'(2)$ and the convexity of $\Gamma(\sigma)$). We see that, while (5.62) is very good in most cases, it poses problems when either σ or ℓ is close to 0.

Let us first deal with the issue of ℓ small. For general α and $\ell\leq 0$,

$$\begin{aligned}|I| &\leq \int_0^\infty e^{-\left(\frac{t^2}{2}\sin 2\alpha - \ell t\cos\left(\frac{\pi}{4}-\alpha\right)+\left(\frac{\pi}{4}-\alpha\right)\tau\right)} t^{\sigma-1}dt\\ &\leq e^{-\left(\frac{\pi}{4}-\alpha\right)\tau}\int_0^\infty e^{-\frac{t^2}{2}\sin 2\alpha} t^\sigma\frac{dt}{t} = \frac{e^{-\left(\frac{\pi}{4}-\alpha\right)\tau}}{(\sin 2\alpha)^{\sigma/2}}\int_0^\infty e^{-\frac{t^2}{2}}t^\sigma\frac{dt}{t}\\ &= \frac{e^{-\left(\frac{\pi}{4}-\alpha\right)\tau}}{(\sin 2\alpha)^{\sigma/2}}\cdot 2^{\sigma/2-1}\int_0^\infty e^{-y}y^{\frac{\sigma}{2}}\frac{dy}{y} = \frac{e^{\alpha\tau}}{2(\sin 2\alpha)^{\sigma/2}}\cdot 2^{\sigma/2}\Gamma(\sigma/2)e^{-\frac{\pi}{4}\tau}.\end{aligned}$$

Here we can choose $\alpha = (\arcsin 2/\tau)/2$ (for $\tau\geq 2$). Then $2\alpha\leq(\pi/2)\cdot(2/\tau) = \pi/\tau$, and so

$$|I| \leq \frac{e^{\frac{\pi}{2\tau}\cdot\tau}}{2(2/\tau)^{\sigma/2}}\cdot 2^{\sigma/2}\Gamma(\sigma/2)e^{-\frac{\pi}{4}\tau} \leq \frac{e^{\pi/2}}{2}\tau^{\sigma/2}\Gamma(\sigma/2)\cdot e^{-\frac{\pi}{4}\tau}. \tag{5.63}$$

The only issue that remains is that σ may be close to 0, in which case $\Gamma(\sigma/2)$ can be large. We can resolve this, as before, by doing an integration by parts. In general, for $-1 < \sigma < 1$, $s \neq 0$:

$$\begin{aligned}|I| &\leq e^{-u^2/2} e(\delta u) \frac{u^s}{s}\Big|_0^{v\infty} - \int_{C'} \left(e^{-u^2/2} e(\delta u)\right)' \frac{u^s}{s} du \\ &= \int_{C'} (u + \ell i) e^{-u^2/2 - \ell i u} \frac{u^s}{s} du \\ &= \frac{1}{s} \int_{C'} e^{-u^2/2} e(\delta u) u^{s+1} du + \frac{\ell i}{s} \int_{C'} e^{-u^2/2} e(\delta u) u^s du.\end{aligned} \tag{5.64}$$

Now we apply (5.62) with $s+1$ and $s+2$ instead of s, and get that

$$\begin{aligned}|I| &= \frac{1}{|s|} \left(\frac{\sqrt{2}}{|\ell|}\right)^{\sigma+2} \Gamma(\sigma+2) \cdot e^{-\frac{\pi}{4}\tau} + \frac{|\ell|}{|s|} \left(\frac{\sqrt{2}}{|\ell|}\right)^{\sigma+1} \Gamma(\sigma+1) \cdot e^{-\frac{\pi}{4}\tau} \\ &\leq \frac{1}{\tau} \left(\frac{\sqrt{2}}{|\ell|}\right)^{\sigma} \left(\frac{4}{\ell^2} + \sqrt{2}\right) e^{-\frac{\pi}{4}\tau}.\end{aligned}$$

Alternatively, we may apply (5.63) and obtain

$$\begin{aligned}|I| &\leq \frac{1}{|s|} \frac{e^{\pi/2}}{2} \Gamma((\sigma+2)/2) \cdot \tau^{(\sigma+2)/2} e^{-\frac{\pi}{4}\tau} + \frac{|\ell|}{|s|} \frac{e^{\pi/2}}{2} \Gamma((\sigma+1)/2) \cdot \tau^{(\sigma+1)/2} e^{-\frac{\pi}{4}\tau} \\ &\leq \frac{e^{\pi/2}\tau^{\sigma/2}}{2} \left(1 + \frac{\sqrt{\pi}|\ell|}{\sqrt{\tau}}\right) e^{-\frac{\pi}{4}\tau}\end{aligned}$$

for $\sigma \in [0,1]$, where we are using the facts that $\Gamma(s) \leq \sqrt{\pi}$ for $s \in [1/2, 1]$ and $\Gamma(s) \leq 1$ for $s \in [1,2]$.

5.4. **Totals.** Summing (5.53) with the bounds obtained in §5.3.6, we obtain our final estimate. Recall that we can reduce the case $\tau < 0$ to the case $\tau > 0$ by reflection. We have proven the following statement.

Proposition 5.1. *Let $f_\delta(t) = e^{-t^2/2} e(\delta t)$, $\delta \in \mathbb{R}$. Let F_δ be the Mellin transform of f_δ, i.e.,*

$$F_\delta(s) = \int_0^\infty e^{-t^2/2} e(\delta t) t^s \frac{dt}{t},$$

where $\delta \in \mathbb{R}$. Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \neq 0$. Let $\ell = -2\pi\delta$. Then, if $\operatorname{sgn}(\delta) \neq \operatorname{sgn}(\tau)$,

$$\begin{aligned}|F_\delta(s)| &\leq C_{0,\tau,\ell} \cdot e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right) \cdot |\tau|} \\ &\quad + C_{1,\tau,\ell} \cdot e^{-0.4798\tau} + C_{2,\tau,\ell} \cdot e^{-\min\left(\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2, \frac{25}{32}|\tau|\right)},\end{aligned} \tag{5.65}$$

where

(5.66)

$$E(\rho) = \frac{1}{2}\left(\arccos \frac{1}{\upsilon(\rho)} - \frac{2(\upsilon(\rho)-1)}{\rho}\right),$$

$$C_{0,\tau,\ell} = \min\left(2, \frac{\sqrt{|\tau|}}{\frac{\frac{2\pi|\delta|}{3.3}}{}}\right)\left(1+\max(7.83^{1-\sigma}, 1.63^{\sigma-1})\right)\left(\frac{3/2}{\min\left(\frac{|\tau|}{2\pi|\delta|}, \sqrt{|\tau|}\right)}\right)^{1-\sigma},$$

$$C_{1,\tau,\ell} = \left(1+\frac{1+\sqrt{2}}{\tau}\right)\tau^{\frac{\sigma}{2}}, \qquad \upsilon(\rho) = \sqrt{\frac{1+\sqrt{1+\rho^2}}{2}},$$

$$C_{2,\tau,\ell} = P_\sigma\left(\min\left(\frac{|\tau|}{2\pi|\delta|}, \frac{5}{4}\sqrt{|\tau|}\right)\right),$$

where $P_\sigma(x) = x^{\sigma-2}$ *if* $\sigma \in [0,2]$, $P_\sigma(x) = x^{\sigma-2} + (\sigma-2)x^{\sigma-4}$ *if* $\sigma \in (2,4]$ *and* $P_\sigma(x) = x^{\sigma-2} + (\sigma-2)x^{\sigma-4} + \ldots + (\sigma-2k)x^{\sigma-2(k+1)}$ *if* $\sigma \in (2k, 2(k+1)]$.

If $\operatorname{sgn}(\delta) = \operatorname{sgn}(\tau)$ *(or* $\delta = 0$*) and* $|\tau| \geq 2$,

$$|F_\delta(s)| \leq C'_{\tau,\ell} e^{-\frac{\pi}{4}|\tau|}, \tag{5.67}$$

where

$$C'_{\tau,\ell} \leq \frac{e^{\pi/2}\tau^{\sigma/2}}{2} \cdot \begin{cases} 1 + \frac{2\pi^{3/2}|\delta|}{\sqrt{|\tau|}} & \text{for } \sigma \in [0,1], \\ \Gamma(\sigma/2) & \text{for } \sigma > 0 \text{ arbitrary.} \end{cases}$$

The terms in (5.65) other than $C_{0,\tau,\ell} \cdot e^{-E(|\tau|/(\pi\delta)^2)|\tau|}$ are usually very small. In practice, we will apply Prop. 5.1 when $|\tau|/2\pi|\delta|$ is larger than a moderate constant (say 8) and $|\tau|$ is larger than a somewhat larger constant (say 100). Thus, $C_{0,\tau,\ell}$ will be bounded.

For comparison, the Mellin transform of $e^{-t^2/2}$ (i.e., $F_0 = Mf_0$) is $2^{s/2-1}\Gamma(s/2)$, which decays like $e^{-(\pi/4)|\tau|}$. For τ very small (e.g., $|\tau| < 2$), it can make sense to use the trivial bound

$$|F_\delta(s)| \leq F_0(\sigma) = \int_0^\infty e^{-t^2/2} t^\sigma \frac{dt}{t} = 2^{\sigma/2-1}\Gamma(\sigma/2) \leq \frac{2^{\sigma/2}}{\sigma} \tag{5.68}$$

for $\sigma \in (0,1]$. Alternatively, we could use integration by parts (much as in (5.64)), followed by the trivial bound:

$$F_\delta(s) = -\int_0^\infty \left(e^{-u^2/2} e(\delta u)\right)' \frac{u^s}{s} du = \frac{F_\delta(s+2)}{s} - \frac{2\pi\delta i}{s} F_\delta(s+1), \tag{5.69}$$

and so

$$|F_\delta(s)| \leq \frac{2^{\frac{\sigma+2}{2}-1}\Gamma\left(\frac{\sigma+2}{2}\right) + 2^{\frac{\sigma+1}{2}-1}|2\pi\delta|\Gamma\left(\frac{\sigma+1}{2}\right)}{|s|} \leq \sqrt{\frac{\pi}{2}} \cdot \frac{1+2\pi|\delta|}{|s|} \tag{5.70}$$

for $0 \leq \sigma \leq 1$, since $2^x\Gamma(x) \leq \sqrt{2\pi}$ for $x \in [1/2, 3/2]$.

It will be useful to have simple approximations to $E(\rho)$ in (5.66).

Lemma 5.2. *Let* $E(\rho)$ *and* $\upsilon(\rho)$ *be as in (5.66). Then*

$$E(\rho) \geq \frac{1}{8}\rho - \frac{5}{384}\rho^3 \tag{5.71}$$

for all $\rho > 0$. *We can also write*

$$E(\rho) = \frac{\pi}{4} - \frac{\beta}{2} - \frac{\sin 2\beta}{4(1+\sin\beta)}, \tag{5.72}$$

where $\beta = \arcsin 1/\upsilon(\rho)$.

Clearly, (5.71) is useful for ρ small, whereas (5.72) is useful for ρ large (since then β is close to 0). Taking derivatives, we see that (5.72) implies that $E(\rho)$ is decreasing on β; thus, $E(\rho)$ is increasing on ρ. Note that (5.71) gives us that

$$E\left(\frac{|\tau|}{(\pi\delta)^2}\right) \cdot |\tau| \geq \frac{1}{2}\left(\frac{\tau}{2\pi\delta}\right)^2 \cdot \left(1 - \frac{5}{48\pi^4}\left(\frac{\tau}{|\delta|^2}\right)^2\right). \tag{5.73}$$

Proof. Let $\alpha = \arccos 1/\upsilon(\rho)$. Then $\upsilon(\rho) = 1/(\cos\alpha)$, whereas

$$\sqrt{1+\rho^2} = 2\upsilon^2(\rho) - 1 = \frac{2}{\cos^2\alpha} - 1,$$

$$\begin{aligned} \rho &= \sqrt{\left(\frac{2}{\cos^2\alpha} - 1\right)^2 - 1} = \sqrt{\frac{4}{\cos^4\alpha} - \frac{4}{\cos^2\alpha}} \\ &= \frac{2\sqrt{1-\cos^2\alpha}}{\cos^2\alpha} = \frac{2\sin\alpha}{\cos^2\alpha}. \end{aligned} \tag{5.74}$$

Thus

(5.75)

$$\begin{aligned} 2E(\rho) &= \alpha - \frac{2\left(\frac{1}{\cos\alpha} - 1\right)}{\frac{2\sin\alpha}{\cos^2\alpha}} = \alpha - \frac{(1-\cos\alpha)\cos\alpha}{\sin\alpha} = \alpha - \frac{(1-\cos^2\alpha)\cos\alpha}{\sin\alpha(1+\cos\alpha)} \\ &= \alpha - \frac{\sin\alpha\cos\alpha}{1+\cos\alpha} = \alpha - \frac{\sin 2\alpha}{4\cos^2\frac{\alpha}{2}}. \end{aligned}$$

By (C.44) and (5.74), this implies that

$$2E(\rho) \geq \frac{\rho}{4} - \frac{5\rho^3}{24\cdot 8},$$

giving us (5.71).

To obtain (5.72), simply define $\beta = \pi/2 - \alpha$; the desired inequality follows from the last two steps of (5.75). □

Let us end by a remark that may be relevant to applications outside number theory. By (5.5), Proposition 5.1 gives us bounds on the parabolic cylinder function $U(a,z)$ for z purely imaginary and $|\Re(a)| \leq 1/2$. The bounds are useful when $|\Im(a)|$ is at least somewhat larger than $|\Im(a)|$ (i.e., when $|\tau|$ is large compared to ℓ). As we have seen in the above, extending the result to broader bands for a is not too hard – integration by parts can be used to push a to the right.

6. Explicit formulas

An *explicit formula* is an expression restating a sum such as $S_{\eta,\chi}(\delta/x, x)$ as a sum of the Mellin transform $G_\delta(s)$ over the zeros of the L function $L(s,\chi)$. More specifically, for us, $G_\delta(s)$ is the Mellin transform of $\eta(s)e(\delta s)$ for some smoothing function η and some $\delta \in \mathbb{R}$. We want a formula whose error terms are good both for δ very close or equal to 0 and for δ farther away from 0. (Indeed, our choices of η will be made so that $F_\delta(s)$ decays rapidly in both cases.)

We will derive two explicit formulas: one for $\eta(t) = t^2 e^{-t^2/(2\sigma)}$, and the other for $\eta = \eta_+$, where η_+ is defined as in (4.7) for some value of $H > 0$ to be set later. As we have already discussed, both functions are variants of $\eta_\heartsuit(t) = e^{-t^2/2}$. Thus, our expressions for G_δ will be based on our study of the Mellin transform F_δ of $\eta_\heartsuit e(\delta t)$ in §5.

6.1. **A general explicit formula.**

Lemma 6.1. *Let $\eta : \mathbb{R}_0^+ \to \mathbb{R}$ be in C^1. Let $x \in \mathbb{R}^+$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$.*

Write $G_\delta(s)$ for the Mellin transform of $\eta(t)e(\delta t)$. Assume that G_δ is holomorphic on $\{s : -1/2 \leq \Re(s) \leq 1+\epsilon\}$. Then

$$\sum_{n=1}^{\infty} \Lambda(n)\chi(n) e\left(\frac{\delta}{x} n\right) \eta(n/x) = I_{q=1} \cdot \widehat{\eta}(-\delta) x - \sum_\rho G_\delta(\rho) x^\rho + O^*(|G_\delta(0)|) + O^*\left((\log q + 6.01) \cdot (|\eta'|_2 + 2\pi|\delta||\eta|_2)\right) x^{-1/2}, \tag{6.1}$$

where

$$I_{q=1} = \begin{cases} 1 & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

and the norms $|\eta|_2$, $|\eta'|_2$ are taken with respect to the usual measure dt. The sum $\sum_\rho$ is a sum over all non-trivial zeros ρ of $L(s,\chi)$.

Proof. Since $G_\delta(s)$ is defined for $\Re(s) \in [-1/2, 1+\epsilon]$,

$$\sum_{n=1}^{\infty} \Lambda(n)\chi(n) e(\delta n/x) \eta(n/x) = \frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} -\frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) x^s ds$$

(see [HL23, Lemma 1] or, e.g., [MV07, p. 144]). We shift the line of integration to $\Re(s) = -1/2$. If $q = 1$ or $\chi(-1) = -1$, then $L(s,\chi)$ does not have a zero at $s = 0$; if $q > 1$ and $\chi(-1) = 1$, then $L(s,\chi)$ has a simple zero at 0 (as discussed in, e.g., [Dav67, §19]). Thus, we obtain

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) x^s ds = I_{q=1} G_\delta(1) x - \sum_\rho G_\delta(\rho) x^\rho - I_{1,\chi} G_\delta(0) - \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) x^s ds, \tag{6.2}$$

where $I_{1,\chi} = 1$ if $q > 1$ and $\chi(-1) = 1$ and $I_{1,\chi} = 0$ otherwise. Of course,

$$G_\delta(1) = M(\eta(t)e(\delta t))(1) = \int_0^\infty \eta(t) e(\delta t) dt = \widehat{\eta}(-\delta).$$

It is time to estimate the integral on the right side of (6.2). By the functional equation (as in, e.g., [IK04, Thm. 4.15]),

$$\frac{L'(s,\chi)}{L(s,\chi)} = \log\frac{\pi}{q} - \frac{1}{2}\psi\left(\frac{s+\kappa}{2}\right) - \frac{1}{2}\psi\left(\frac{1-s+\kappa}{2}\right) - \frac{L'(1-s,\overline{\chi})}{L(1-s,\overline{\chi})}, \tag{6.3}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ and

$$\kappa = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

By $\psi(1-x) - \psi(x) = \pi \cot \pi x$ (immediate from $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$) and $\psi(s) + \psi(s+1/2) = 2(\psi(2s) - \log 2)$ (Legendre),

$$-\frac{1}{2}\left(\psi\left(\frac{s+\kappa}{2}\right) + \psi\left(\frac{1-s+\kappa}{2}\right)\right) = -\psi(1-s) + \log 2 + \frac{\pi}{2}\cot\frac{\pi(s+\kappa)}{2}. \tag{6.4}$$

Now, if $\Re(z) = 3/2$, then $|t^2 + z^2| \geq 9/4$ for all real t. Hence, by [OLBC10, (5.9.15)] and [GR00, (3.411.1)],

$$
\begin{aligned}
\psi(z) &= \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t dt}{(t^2+z^2)(e^{2\pi t}-1)} \\
&= \log z - \frac{1}{2z} + 2\cdot O^*\left(\int_0^\infty \frac{t dt}{\frac{9}{4}(e^{2\pi t}-1)}\right) \\
&= \log z - \frac{1}{2z} + \frac{8}{9} O^*\left(\int_0^\infty \frac{t dt}{e^{2\pi t}-1}\right) \\
&= \log z - \frac{1}{2z} + \frac{8}{9}\cdot O^*\left(\frac{1}{(2\pi)^2}\Gamma(2)\zeta(2)\right) \\
&= \log z - \frac{1}{2z} + O^*\left(\frac{1}{27}\right) = \log z + O^*\left(\frac{10}{27}\right).
\end{aligned} \tag{6.5}
$$

Thus, in particular, $\psi(1-s) = \log(3/2 - i\tau) + O^*(10/27)$, where we write $s = 1/2 + i\tau$. Now

$$\left|\cot\frac{\pi(s+\kappa)}{2}\right| = \left|\frac{e^{\mp\frac{\pi}{4}i - \frac{\pi}{2}\tau} + e^{\pm\frac{\pi}{4}i + \frac{\pi}{2}\tau}}{e^{\mp\frac{\pi}{4}i - \frac{\pi}{2}\tau} - e^{\pm\frac{\pi}{4}i + \frac{\pi}{2}\tau}}\right| = 1.$$

Since $\Re(s) = -1/2$, a comparison of Dirichlet series gives

$$\left|\frac{L'(1-s,\overline{\chi})}{L(1-s,\overline{\chi})}\right| \leq \frac{|\zeta'(3/2)|}{|\zeta(3/2)|} \leq 1.50524, \tag{6.6}$$

where $\zeta'(3/2)$ and $\zeta(3/2)$ can be evaluated by Euler-Maclaurin. Therefore, (6.3) and (6.4) give us that, for $s = -1/2 + i\tau$,

$$
\begin{aligned}
\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| &\leq \left|\log\frac{q}{\pi}\right| + \log\left|\frac{3}{2} + i\tau\right| + \frac{10}{27} + \log 2 + \frac{\pi}{2} + 1.50524 \\
&\leq \left|\log\frac{q}{\pi}\right| + \frac{1}{2}\log\left(\tau^2 + \frac{9}{4}\right) + 4.1396.
\end{aligned} \tag{6.7}
$$

Recall that we must bound the integral on the right side of (6.2). The absolute value of the integral is at most $x^{-1/2}$ times

$$\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left|\frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s)\right| ds. \tag{6.8}$$

By Cauchy-Schwarz, this is at most

$$\sqrt{\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left|\frac{L'(s,\chi)}{L(s,\chi)}\cdot\frac{1}{s}\right|^2 |ds|}\cdot\sqrt{\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |G_\delta(s)s|^2 |ds|}$$

By (6.7),

$$
\begin{aligned}
\sqrt{\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left|\frac{L'(s,\chi)}{L(s,\chi)}\cdot\frac{1}{s}\right|^2 |ds|} &\leq \sqrt{\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left|\frac{\log q}{s}\right|^2 |ds|} \\
&\quad + \sqrt{\int_{-\infty}^{\infty} \frac{\left|\frac{1}{2}\log\left(\tau^2+\frac{9}{4}\right) + 4.1396 + \log\pi\right|^2}{\frac{1}{4}+\tau^2} d\tau} \\
&\leq \sqrt{2\pi}\log q + \sqrt{226.844},
\end{aligned}
$$

where we compute the last integral numerically.[8]

By (2.5), $G_\delta(s)s$ is the Mellin transform of

$$-t\frac{d(e(\delta t)\eta(t))}{dt} = -2\pi i\delta t e(\delta t)\eta(t) - te(\delta t)\eta'(t) \tag{6.9}$$

Hence, by Plancherel (as in (2.4)),

$$\sqrt{\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |G_\delta(s)s|^2 |ds|} = \sqrt{\int_0^\infty |-2\pi i\delta t e(\delta t)\eta(t) - te(\delta t)\eta'(t)|^2 t^{-2} dt} \tag{6.10}$$
$$= 2\pi|\delta|\sqrt{\int_0^\infty |\eta(t)|^2 dt} + \sqrt{\int_0^\infty |\eta'(t)|^2 dt}.$$

Thus, (6.8) is at most

$$\left(\log q + \sqrt{\frac{226.844}{2\pi}}\right)\cdot\left(|\eta'|_2 + 2\pi|\delta||\eta|_2\right).$$

□

It now remains to bound the sum $\sum_\rho G_\delta(\rho)x^\rho$ in (6.1). Clearly

$$\left|\sum_\rho G_\delta(\rho)x^\rho\right| \leq \sum_\rho |G_\delta(\rho)|\cdot x^{\Re(\rho)}.$$

Recall that these are sums over the non-trivial zeros ρ of $L(s,\chi)$.

We first prove a general lemma on sums of values of functions on the non-trivial zeros of $L(s,\chi)$.

Lemma 6.2. *Let $f:\mathbb{R}^+\to\mathbb{C}$ be piecewise C^1. Assume $\lim_{t\to\infty} f(t)t\log t = 0$. Then, for any $y \geq 1$,*

$$\sum_{\substack{\rho \text{ non-trivial} \\ \Im(\rho)>y}} f(\Im(\rho)) = \frac{1}{2\pi}\int_y^\infty f(T)\log\frac{qT}{2\pi}dT \tag{6.11}$$
$$+ \frac{1}{2}O^*\left(|f(y)|g_\chi(y) + \int_y^\infty |f'(T)|\cdot g_\chi(T)dT\right),$$

where

$$g_\chi(T) = 0.5\log qT + 17.7 \tag{6.12}$$

If f is real-valued and decreasing on $[y,\infty)$, the second line of (6.11) equals

$$O^*\left(\frac{1}{4}\int_y^\infty \frac{f(T)}{T}dT\right).$$

Proof. Write $N(T,\chi)$ for the number of non-trivial zeros of $L(s,\chi)$ with $|\Im(s)| \leq T$. Write $N^+(T,\chi)$ for the number of (necessarily non-trivial) zeros of $L(s,\chi)$

[8] By a rigorous integration from $\tau = -100000$ to $\tau = 100000$ using VNODE-LP [Ned06], which runs on the PROFIL/BIAS interval arithmetic package[Knü99].

with $0 < \Im(s) \leq T$. Then, for any $f : \mathbb{R}^+ \to \mathbb{C}$ with f piecewise differentiable and $\lim_{t\to\infty} f(t)N(T,\chi) = 0$,

$$\begin{aligned}\sum_{\rho:\Im(\rho)>y} f(\Im(\rho)) &= \int_y^\infty f(T)\, dN^+(T,\chi)\\ &= -\int_y^\infty f'(T)(N^+(T,\chi) - N^+(y,\chi))dT\\ &= -\frac{1}{2}\int_y^\infty f'(T)(N(T,\chi) - N(y,\chi))dT.\end{aligned}$$

Now, by [Ros41, Thms. 17–19] and [McC84, Thm. 2.1] (see also [Tru, Thm. 1]),

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi e} + O^*(g_\chi(T)) \tag{6.13}$$

for $T \geq 1$, where $g_\chi(T)$ is as in (6.12). (This is a classical formula; the references serve to prove the explicit form (6.12) for the error term $g_\chi(T)$.)

Thus, for $y \geq 1$,

$$\begin{aligned}\sum_{\rho:\Im(\rho)>y} f(\Im(\rho)) &= -\frac{1}{2}\int_y^\infty f'(T)\left(\frac{T}{\pi}\log\frac{qT}{2\pi e} - \frac{y}{\pi}\log\frac{qy}{2\pi e}\right)dT\\ &\quad+\frac{1}{2}O^*\left(|f(y)|g_\chi(y) + \int_y^\infty \left|f'(T)\right|\cdot g_\chi(T)dT\right).\end{aligned} \tag{6.14}$$

Here

$$-\frac{1}{2}\int_y^\infty f'(T)\left(\frac{T}{\pi}\log\frac{qT}{2\pi e} - \frac{y}{\pi}\log\frac{qy}{2\pi e}\right)dT = \frac{1}{2\pi}\int_y^\infty f(T)\log\frac{qT}{2\pi}dT. \tag{6.15}$$

If f is real-valued and decreasing (and so, by $\lim_{t\to\infty} f(t) = 0$, non-negative),

$$\begin{aligned}|f(y)|g_\chi(y) + \int_y^\infty \left|f'(T)\right|\cdot g_\chi(T)dT &= f(y)g_\chi(y) - \int_y^\infty f'(T)g_\chi(T)dT\\ &= 0.5\int_y^\infty \frac{f(T)}{T}dT,\end{aligned}$$

since $g_\chi'(T) \leq 0.5/T$ for all $T \geq T_0$. □

Lemma 6.3. *Let $\eta : \mathbb{R}_0^+ \to \mathbb{R}$ be such that both $\eta(t)$ and $(\log t)\eta(t)$ lie in $L_1 \cap L_2$ (with respect to dt). Let $\delta \in \mathbb{R}$. Let $G_\delta(s)$ be the Mellin transform of $\eta(t)e(\delta t)$.*

Let χ be a primitive character mod q, $q \geq 1$. Let $T_0 \geq 1$. Assume that all non-trivial zeros ρ of $L(s,\chi)$ with $|\Im(\rho)| \leq T_0$ lie on the critical line. Then

$$\sum_{\substack{\rho \text{ non-trivial}\\ |\Im(\rho)|\leq T_0}} |G_\delta(\rho)|$$

is at most

$$\begin{aligned}&(|\eta|_2 + |\eta\cdot\log|_2)\sqrt{T_0}\log qT_0 + (17.21|\eta\cdot\log|_2 - (\log 2\pi\sqrt{e})|\eta|_2)\sqrt{T_0}\\ &\quad+\left|\eta(t)/\sqrt{t}\right|_1\cdot(1.32\log q + 34.5)\end{aligned}$$

Proof. For $s = 1/2 + i\tau$, we have the trivial bound

$$|G_\delta(s)| \leq \int_0^\infty |\eta(t)|t^{1/2}\frac{dt}{t} = \left|\eta(t)/\sqrt{t}\right|_1, \tag{6.16}$$

where F_δ is as in (6.19). We also have the trivial bound
(6.17)

$$|G_\delta'(s)| = \left|\int_0^\infty (\log t)\eta(t) t^s \frac{dt}{t}\right| \leq \int_0^\infty |(\log t)\eta(t)| t^\sigma \frac{dt}{t} = \left|(\log t)\eta(t) t^{\sigma-1}\right|_1$$

for $s = \sigma + i\tau$.

Let us start by bounding the contribution of very low-lying zeros ($|\Im(\rho)| \leq 1$). By (6.13) and (6.12),

$$N(1,\chi) = \frac{1}{\pi}\log\frac{q}{2\pi e} + O^*(0.5\log q + 17.7) = O^*(0.819\log q + 16.8).$$

Therefore,

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|\leq 1}} |G_\delta(\rho)| \leq \left|\eta(t) t^{-1/2}\right|_1 \cdot (0.819\log q + 16.8).$$

Let us now consider zeros ρ with $|\Im(\rho)| > 1$. Apply Lemma 6.2 with $y = 1$ and

$$f(t) = \begin{cases} |G_\delta(1/2 + it)| & \text{if } t \leq T_0, \\ 0 & \text{if } t > T_0. \end{cases}$$

This gives us that

$$\sum_{\rho: 1 < |\Im(\rho)| \leq T_0} f(\Im(\rho)) = \frac{1}{\pi}\int_1^{T_0} f(T)\log\frac{qT}{2\pi} dT + O^*\left(|f(1)| g_\chi(1) + \int_1^\infty |f'(T)| \cdot g_\chi(T)\, dT\right), \tag{6.18}$$

where we are using the fact that $f(\sigma + i\tau) = f(\sigma - i\tau)$ (because η is real-valued). By Cauchy-Schwarz,

$$\frac{1}{\pi}\int_1^{T_0} f(T)\log\frac{qT}{2\pi} dT \leq \sqrt{\frac{1}{\pi}\int_1^{T_0} |f(T)|^2 dT} \cdot \sqrt{\frac{1}{\pi}\int_1^{T_0} \left(\log\frac{qT}{2\pi}\right)^2 dT}.$$

Now

$$\frac{1}{\pi}\int_1^{T_0} |f(T)|^2 dT \leq \frac{1}{2\pi}\int_{-\infty}^\infty \left|G_\delta\left(\frac{1}{2} + iT\right)\right|^2 dT \leq \int_0^\infty |e(\delta t)\eta(t)|^2 dt = |\eta|_2^2$$

by Plancherel (as in (2.4)). We also have

$$\int_1^{T_0} \left(\log\frac{qT}{2\pi}\right)^2 dT \leq \frac{2\pi}{q}\int_0^{\frac{qT_0}{2\pi}} (\log t)^2 dt \leq \left(\left(\log\frac{qT_0}{2\pi e}\right)^2 + 1\right) \cdot T_0.$$

Hence

$$\frac{1}{\pi}\int_1^{T_0} f(T)\log\frac{qT}{2\pi} dT \leq \sqrt{\left(\log\frac{qT_0}{2\pi e}\right)^2 + 1} \cdot |\eta|_2 \sqrt{T_0}.$$

Again by Cauchy-Schwarz,

$$\int_1^\infty |f'(T)| \cdot g_\chi(T)\, dT \leq \sqrt{\frac{1}{2\pi}\int_{-\infty}^\infty |f'(T)|^2 dT} \cdot \sqrt{\frac{1}{\pi}\int_1^{T_0} |g_\chi(T)|^2 dT}.$$

Since $|f'(T)| = |G'_\delta(1/2+iT)|$ and $(M\eta)'(s)$ is the Mellin transform of $\log(t)\cdot e(\delta t)\eta(t)$ (by (2.5)),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f'(T)|^2 dT = |\eta(t)\log(t)|_2.$$

Much as before,

$$\begin{aligned}\int_1^{T_0} |g_\chi(T)|^2 dT &\leq \int_0^{T_0} (0.5\log qT + 17.7)^2 dT\\ &= (0.25(\log qT_0)^2 + 17.2(\log qT_0) + 296.09)T_0.\end{aligned}$$

Summing, we obtain

$$\begin{aligned}&\frac{1}{\pi}\int_1^{T_0} f(T)\log\frac{qT}{2\pi}dT + \int_1^{\infty} |f'(T)|\cdot g_\chi(T)\, dT\\ &\quad\leq \left(\left(\log\frac{qT_0}{2\pi e}+\frac{1}{2}\right)|\eta|_2 + \left(\frac{\log qT_0}{2}+17.21\right)|\eta(t)(\log t)|_2\right)\sqrt{T_0}\end{aligned}$$

Finally, by (6.16) and (6.12),

$$|f(1)|g_\chi(1) \leq \left|\eta(t)/\sqrt{t}\right|_1 \cdot (0.5\log q + 17.7).$$

By (6.18) and the assumption that all non-trivial zeros with $|\Im(\rho)| \leq T_0$ lie on the line $\Re(s) = 1/2$, we conclude that

$$\begin{aligned}\sum_{\substack{\rho \text{ non-trivial}\\ 1<|\Im(\rho)|\leq T_0}} |G_\delta(\rho)| &\leq (|\eta|_2 + |\eta\cdot\log|_2)\sqrt{T_0}\log qT_0\\ &\quad + (17.21|\eta\cdot\log|_2 - (\log 2\pi\sqrt{e})|\eta|_2)\sqrt{T_0}\\ &\quad + \left|\eta(t)/\sqrt{t}\right|_1 \cdot (0.5\log q + 17.7).\end{aligned}$$

□

6.2. **Sums and decay for $\eta(t) = t^2 e^{-t^2/2}$ and $\eta^*(t)$.** Let

$$\begin{aligned}\eta(t) &= \begin{cases} t^2 e^{-t^2/2} & \text{if } t \geq 0,\\ 0 & \text{if } t < 0,\end{cases}\\ F_\delta(s) &= (M(e^{-\frac{t^2}{2}}e(\delta t)))(s),\\ G_\delta(s) &= (M(\eta(t)e(\delta t)))(s).\end{aligned} \tag{6.19}$$

Then, by the definition of the Mellin transform,

$$G_\delta(s) = F_\delta(s+2).$$

Hence

$$|G_\delta(0)| \leq |(M\eta)(0)| = \int_0^\infty te^{-t^2/2}dt = 1$$

and

$$|\eta|_2^2 = \frac{3}{8}\sqrt{\pi}, \qquad |\eta'|_2^2 = \frac{7}{16}\sqrt{\pi},$$

$$|\eta\cdot\log|_2^2 \leq 0.16364, \qquad |\eta(t)/\sqrt{t}|_1 = \frac{2^{1/4}\Gamma(1/4)}{4} \leq 1.07791.$$

Lemma 6.4. *Let $\eta(t) = t^2 e^{-t^2/2}$. Let $x \in \mathbb{R}^+$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$. Assume that all non-trivial zeros ρ of $L(s,\chi)$ with $|\Im(\rho)| \leq T_0$ satisfy $\Re(s) = 1/2$. Assume that $T_0 \geq \max(4\pi^2|\delta|, 100)$.*

Write $G_\delta(s)$ for the Mellin transform of $\eta(t)e(\delta t)$. Then

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |G_\delta(\rho)| \leq T_0 \log \frac{qT_0}{2\pi} \cdot \left(3.5 e^{-0.1598 T_0} + 2.56 e^{-0.1065 \cdot \frac{T_0^2}{(\pi\delta)^2}}\right).$$

Here we have preferred a bound with a simple form. It is probably feasible to derive from the results in §5 a bound essentially proportional to $e^{-E(\rho)T_0}$, where $\rho = T_0/(\pi\delta)^2$ and $E(\rho)$ is as in (5.66). (This would behave as $e^{-(\pi/4)\tau}$ for ρ large and as $e^{-0.125(T_0/(\pi\delta))^2}$ for ρ small.)

Proof. First of all,

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |G_\delta(\rho)| = \sum_{\substack{\rho \text{ non-trivial} \\ \Im(\rho)>T_0}} \left(|F_\delta(\rho+2)| + |F_\delta((1-\rho)+2)|\right),$$

where we are using $G_\delta(\rho) = F_\delta(\rho+2)$ and the functional equation (which implies that non-trivial zeros come in pairs ρ, $1-\rho$).

By Prop. 5.1, given $\rho = \sigma + i\tau$, $\sigma \in [0,1]$, $\tau \geq \max(1, 2\pi|\delta|)$, we obtain that

$$|F_\delta(\rho+2)| + |F_\delta((1-\rho)+2)|$$

is at most

$$C_{0,\tau,\ell} \cdot e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right)\cdot\tau} + C_{1,\tau} \cdot e^{-0.4798\tau} + C_{2,\tau} \cdot e^{-\min\left(\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2, \frac{25}{32}|\tau|\right)} + C'_\tau e^{-\frac{\pi}{4}|\tau|},$$

where $E(\rho)$ is as in (5.66),

$$C_{0,\tau,\ell} \leq 2 \cdot (1+1.63^2) \cdot \left(\frac{\min\left(\frac{|\tau|}{2\pi|\delta|}, \sqrt{|\tau|}\right)}{3/2}\right)^2 \leq 3.251 \min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2, |\tau|\right)$$

$$C_{1,\tau} \leq \left(1 + \frac{1+\sqrt{2}}{|\tau|}\right)|\tau|^{\frac{3}{2}}, \quad C_{2,\tau} \leq \min\left(\frac{|\tau|}{|\ell|}, \frac{5}{4}\sqrt{|\tau|}\right) + 1, \quad C'_\tau \leq \frac{e^{\pi/2}|\tau|^{3/2}}{2}$$

and $\ell = -2\pi\delta$.

For $\tau \geq T_0 \geq 100$,

$$\left(1 + \frac{1+\sqrt{2}}{|\tau|}\right) + \frac{e^{\pi/2}}{2} \cdot e^{-\left(\frac{\pi}{4} - 0.4798\right)|\tau|} \leq 1.025 \tag{6.20}$$

and so

$$C_{1,\tau} \cdot e^{-0.4798|\tau|} + C'_\tau e^{-\frac{\pi}{4}|\tau|} \leq 1.025|\tau|^{3/2} e^{-0.4798|\tau|}.$$

It is clear that this is a decreasing function of $|\tau|$ for $|\tau| > 3/(2 \cdot 0.4798)$ (and hence for $|\tau| \geq 100$).

We bound

$$E(\rho) \geq \begin{cases} E(1.5) \geq 0.1598 & \text{if } \rho \geq 1.5, \\ \frac{E(1.5)}{1.5}\rho \geq 0.1065\rho & \text{if } \rho < 1.5. \end{cases} \tag{6.21}$$

This holds for $\rho \geq 1.5$ because $E(\rho)$ is increasing on ρ, for $\rho \leq 1.19$ because of (5.71), and for $\rho \in [1.19, 1.5]$ by the bisection method (with 20 iterations).

The bound (6.21) implies immediately that

$$e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right)\cdot\tau} \le e^{-0.1598\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right)} \tag{6.22}$$

Since $\tau \ge T_0 \ge \max(4\pi^2|\delta|, 100)$,

$$\begin{aligned} e^{-\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2} &\le 0.055e^{-0.1598\cdot\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2}, \\ e^{-\frac{25}{32}\tau} &\le 1.1\cdot 10^{-28}e^{-0.1598\tau}, \end{aligned} \tag{6.23}$$

and

$$0.055\cdot\left(\frac{|\tau|}{|\ell|}+1\right) \le \frac{0.055}{4\pi}\left(\frac{1}{2}+\frac{1}{4\pi}\right)\cdot\left(\frac{|\tau|}{\pi\delta}\right)^2 \le 0.0039\cdot\frac{2}{3}\left(\frac{|\tau|}{\pi\delta}\right)^2$$
$$0.055\cdot\left(\frac{5}{4}\sqrt{|\tau|}+1\right) \le 0.055\left(\frac{5}{4}\cdot\frac{1}{10}+\frac{1}{100}\right)|\tau| \le 0.0075|\tau|.$$

Hence
(6.24)

$$C_{0,\tau,\ell}\cdot e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right)\cdot\tau} + C_{2,\tau}\cdot e^{-\min\left(\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2,\frac{25}{32}|\tau|\right)} \le C'_{0,\tau,\ell}\cdot e^{-0.1598\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right)},$$

where

$$C'_{0,\tau,\delta} = 3.26\cdot\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right).$$

It is clear that the right side of (6.24) is a decreasing function of τ when

$$\min\left(\left(\frac{2}{3}\frac{\tau}{\pi\delta}\right)^2,|\tau|\right) \ge \frac{1}{0.1598}$$

(and hence when $\tau \ge T_0 \ge \max(4\pi^2|\delta|, 100)$).

We thus have

$$\sum_{\substack{\rho \text{ non-trivial}\\ |\Im(\rho)|>T_0}} |G_\delta(\rho)| \le \sum_{\substack{\rho \text{ non-trivial}\\ \Im(\rho)>T_0}} f(\Im(\rho)),$$

where

$$f(\tau) = C'_{0,\tau,\delta}\cdot e^{-0.1598\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right)} + 1.025|\tau|^{3/2}e^{-0.478|\tau|} \tag{6.25}$$

is a decreasing function of τ for $\tau \ge T_0$.

We can now apply Lemma 6.2. We obtain that

$$\sum_{\substack{\rho \text{ non-trivial}\\ \Im(\rho)>T_0}} f(\Im(\rho)) \le \int_{T_0}^{\infty} f(T)\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)dT.$$

If $|\delta| \le 4$, then the condition $\tau \ge T_0 \ge 4\pi^2|\delta|$ implies $\tau \ge (\pi\delta)^2$, and so $\min((\tau/\pi\delta)^2, |\tau|) = |\tau|$. In that case, the contribution of the term in (6.25) involving $C'_{0,\tau,\ell}$ is at most

$$3.26\cdot\int_{T_0}^{\infty}\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)Te^{-0.1598T}dT \tag{6.26}$$

If $|\delta| > 4$, the contribution of $C'_{0,\tau,\ell}$ is at most

$$3.26\cdot\int_{\max(T_0,(\pi\delta)^2)}^{\infty}\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)Te^{-0.1598T}dT \tag{6.27}$$

plus (if $T_0 < (\pi\delta)^2$)

$$(6.28)\quad \begin{aligned} &3.26 \cdot \int_{T_0}^{(\pi\delta)^2} \left(\frac{1}{2\pi}\log\frac{qT}{2\pi} + \frac{1}{4T}\right) \frac{T^2}{\pi^2\delta^2} e^{-0.1598\cdot\frac{2}{3}\frac{T^2}{(\pi\delta)^2}} dT \\ &\leq 3.26\pi|\delta| \cdot \int_{\frac{T_0}{\pi|\delta|}}^{\infty} \left(\frac{1}{2\pi}\log\frac{q|\delta|t}{2} + \frac{1}{4\pi|\delta|t}\right) t^2 e^{-0.1065t^2} dt. \end{aligned}$$

For any $y \geq 1$, $c, c_1 > 0$,

$$\int_y^\infty t^2 e^{-ct^2} dt < \int_y^\infty \left(t^2 + \frac{1}{4c^2t^2}\right) e^{-ct^2} dt = \left(\frac{y}{2c} + \frac{1}{4c^2y}\right) \cdot e^{-cy^2},$$

$$\begin{aligned} \int_y^\infty (t^2\log t + c_1 t) \cdot e^{-ct^2} dt &\leq \int_y^\infty \left(t^2 \log t + \frac{at\log et}{2c} - \frac{\log et}{2c} - \frac{a}{4c^2t}\right) e^{-ct^2} dt \\ &= \frac{(2cy+a)\log y + a}{4c^2} \cdot e^{-cy^2}, \end{aligned}$$

where

$$a = \frac{c_1 y + \frac{\log ey}{2c}}{\frac{y\log ey}{2c} - \frac{1}{4c^2y}} = \frac{1}{y} + \frac{c_1 y + \frac{1}{4c^2y^2}}{\frac{y\log ey}{2c} - \frac{1}{4c^2y}}.$$

Setting $c = 0.1065$, $c_1 = 1/(2|\delta|) \leq 8$ and $y = T_0/(\pi|\delta|) \geq 4\pi$, we obtain

$$\begin{aligned} &\int_{\frac{T_0}{\pi|\delta|}}^\infty \left(\frac{1}{2\pi}\log\frac{q|\delta|t}{2} + \frac{1}{4\pi|\delta|t}\right) t^2 e^{-0.1065t^2} dt \\ &\leq \left(\frac{1}{2\pi}\log\frac{q|\delta|}{2}\right) \cdot \left(\frac{T_0}{2\pi c|\delta|} + \frac{1}{4c^2\cdot 4\pi}\right) \cdot e^{-0.1065\left(\frac{T_0}{\pi|\delta|}\right)^2} \\ &+ \frac{1}{2\pi} \cdot \frac{\left(2c\frac{T_0}{\pi|\delta|} + a\right)\log\frac{T_0}{\pi|\delta|} + a}{4c^2} \cdot e^{-0.1065\left(\frac{T_0}{\pi|\delta|}\right)^2} \end{aligned}$$

and

$$a \leq \frac{1}{4\pi} + \frac{\frac{4\pi}{8} + \frac{1}{4\cdot 0.1065^2\cdot(4\pi)^2}}{\frac{4\pi\log 4\pi e}{2\cdot 0.1065} - \frac{1}{4\cdot 0.1065^2\cdot 4\pi}} \leq 0.088.$$

Multiplying by $3.26\pi|\delta|$, we get that (6.28) is at most $e^{-0.1065\left(\frac{T_0}{\pi|\delta|}\right)^2}$ times

$$(6.29)\quad \begin{aligned} &\left((2.44T_0 + 2.86|\delta|)\cdot\log\frac{q|\delta|}{2} + 2.44T_0\log\frac{T_0}{\pi|\delta|} + 3.2|\delta|\log\frac{eT_0}{\pi|\delta|}\right) \\ &\leq \left(2.44 + 3.2\cdot\frac{1+\frac{1}{\log T_0/\pi|\delta|}}{T_0/|\delta|}\right) T_0\log\frac{qT_0}{2\pi} \leq 2.56T_0\log\frac{qT_0}{2\pi}, \end{aligned}$$

where we are using several times the assumption that $T_0 \geq 4\pi^2|\delta|$.

Let us now go back to (6.26). For any $y \geq 1$, $c, c_1 > 0$,

$$\int_y^\infty te^{-ct}dt = \left(\frac{y}{c} + \frac{1}{c^2}\right) e^{-cy},$$

$$\begin{aligned} \int_y^\infty \left(t\log t + \frac{c_1}{t}\right) e^{-ct} dt &\leq \int_y^\infty \left(\left(t + \frac{a-1}{c}\right)\log t - \frac{1}{c} - \frac{a}{c^2t}\right) e^{-ct} dt \\ &\leq \left(\frac{y}{c} + \frac{a}{c^2}\right) e^{-cy}\log y, \end{aligned}$$

where

$$a = \frac{\frac{\log y}{c} + \frac{1}{c} + \frac{c_1}{y}}{\frac{\log y}{c} - \frac{1}{c^2 y}}.$$

Setting $c = 0.1598$, $c_1 = \pi/2$, $y = T_0 \geq 100$, we obtain that

$$\int_{T_0}^{\infty} \left(\frac{1}{2\pi} \log \frac{qT}{2\pi} + \frac{1}{4T}\right) T e^{-0.1598 T} dT$$
$$\leq \frac{1}{2\pi} \left(\log \frac{q}{2\pi} \cdot \left(\frac{T_0}{c} + \frac{1}{c^2}\right) + \left(\frac{T_0}{c} + \frac{a}{c^2}\right) \log T_0\right) e^{-0.1598 T_0}$$

and

$$a \leq \frac{\frac{\log T_0}{0.1598} + \frac{1}{0.1598} + \frac{\pi/2}{T_0}}{\frac{\log T_0}{0.1598} - \frac{1}{0.1598^2 T_0}} \leq 1.235.$$

Multiplying by 3.26 and simplifying, we obtain that (6.26) is at most

$$3.5 T_0 \log \frac{qT_0}{2\pi} \cdot e^{-0.1148 T_0}. \tag{6.30}$$

Obviously, (6.27) is bounded above by (6.26), and hence it is bounded above by (6.30) as well. □

Proposition 6.5. *Let $\eta(t) = t^2 e^{-t^2/2}$. Let $x \in \mathbb{R}^+$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$. Assume that all non-trivial zeros ρ of $L(s, \chi)$ with $|\Im(\rho)| \leq T_0$ lie on the critical line. Assume that $T_0 \geq \max(4\pi^2 |\delta|, 100)$.*

Then

(6.31)

$$\sum_{n=1}^{\infty} \Lambda(n) \chi(n) e\left(\frac{\delta}{x} n\right) \eta(n/x) = \begin{cases} \widehat{\eta}(-\delta) x + O^*\left(\mathrm{err}_{\eta,\chi}(\delta, x)\right) \cdot x & \text{if } q = 1, \\ O^*\left(\mathrm{err}_{\eta,\chi}(\delta, x)\right) \cdot x & \text{if } q > 1, \end{cases}$$

where

$$\mathrm{err}_{\eta,\chi}(\delta, x) = T_0 \log \frac{qT_0}{2\pi} \cdot \left(3.5 e^{-0.1598 T_0} + 2.56 e^{-0.1065 \cdot \frac{T_0^2}{(\pi\delta)^2}}\right)$$
$$+ \left(1.22 \sqrt{T_0} \log qT_0 + 5.056 \sqrt{T_0} + 1.423 \log q + 38.19\right) \cdot x^{-1/2}$$
$$+ (\log q + 6.01) \cdot (0.89 + 5.13|\delta|) \cdot x^{-3/2}.$$

Proof. Immediate from Lemma 6.1, Lemma 6.3 and Lemma 6.4. □

Now that we have Prop. 6.5, we can derive from it similar bounds for a smoothing defined as the multiplicative convolution of η with something else – just as we discussed at the beginning of §4.3.

Corollary 6.6. *Let $\eta(t) = t^2 e^{-t^2/2}$, $\eta_1 = 2 \cdot I_{[1/2,1]}$, $\eta_2 = \eta_1 *_M \eta_1$. Let $\eta_* = \eta_2 *_M \eta$. Let $x \in \mathbb{R}^+$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$. Assume that all non-trivial zeros ρ of $L(s, \chi)$ with $|\Im(\rho)| \leq T_0$ lie on the critical line. Assume that $T_0 \geq \max(4\pi^2 |\delta|, 100)$.*

Then

(6.32)

$$\sum_{n=1}^{\infty} \Lambda(n) \chi(n) e\left(\frac{\delta}{x} n\right) \eta_*(n/x) = \begin{cases} \widehat{\eta_*}(-\delta) x + O^*\left(\mathrm{err}_{\eta_*,\chi}(\delta, x)\right) \cdot x & \text{if } q = 1, \\ O^*\left(\mathrm{err}_{\eta_*,\chi}(\delta, x)\right) \cdot x & \text{if } q > 1, \end{cases}$$

where
(6.33)
$$\begin{aligned}\mathrm{err}_{\eta,\chi_*}(\delta,x) &= T_0 \log\frac{qT_0}{2\pi}\cdot\left(3.5e^{-0.1598T_0} + 0.0074\cdot e^{-0.1065\cdot\frac{T_0^2}{(\pi\delta)^2}}\right)\\ &\quad+\left(1.22\sqrt{T_0}\log qT_0 + 5.056\sqrt{T_0} + 1.423\log q + 38.19\right)\cdot x^{-\frac{1}{2}}\\ &\quad+(\log q + 6.01)\cdot(0.89+2.7|\delta|)\cdot x^{-3/2}.\end{aligned}$$

Proof. The left side of (6.32) equals

$$\int_0^\infty \sum_{n=1}^\infty \Lambda(n)\chi(n)e\left(\frac{\delta n}{x}\right)\eta\left(\frac{n}{wx}\right)\eta_2(w)\frac{dw}{w}$$

$$= \int_{\frac{1}{4}}^1 \sum_{n=1}^\infty \Lambda(n)\chi(n)e\left(\frac{\delta wn}{wx}\right)\eta\left(\frac{n}{wx}\right)\eta_2(w)\frac{dw}{w},$$

since η_2 is supported on $[-1/4,1]$. By Prop. 6.5, the main term (if $q=1$) contributes

$$\begin{aligned}&\int_{\frac{1}{4}}^1 \widehat{\eta}(-\delta w)xw\cdot\eta_2(w)\frac{dw}{w} = x\int_0^\infty \widehat{\eta}(-\delta w)\eta_2(w)dw\\ &= x\int_0^\infty\int_{-\infty}^\infty \eta(t)e(\delta wt)dt\eta_2(w)dw = x\int_0^\infty\int_{-\infty}^\infty \eta\left(\frac{r}{w}\right)e(\delta r)\frac{dt}{w}\eta_2(w)dw\\ &= x\int_{-\infty}^\infty\left(\int_0^\infty \eta\left(\frac{r}{w}\right)\eta_2(w)\frac{dw}{w}\right)e(\delta r)dr = \widehat{\eta_*}(-\delta)\cdot x.\end{aligned}$$

The error term is

$$\int_{\frac{1}{4}}^1 \mathrm{err}_{\eta,\chi}(\delta w, xw)\cdot wx\cdot\eta_2(w)\frac{dw}{w} = x\cdot\int_{\frac{1}{4}}^1 \mathrm{err}_{\eta,\chi}(\delta w,xw)\eta_2(w)dw. \tag{6.34}$$

Since $\int_w \eta_2(w)dw = 1$, $\int_w w\eta_2(w)dw = (3/4)\log 2$ and[9]

$$\int_w e^{-0.1065\cdot(4\pi)^2\left(\frac{1}{w^2}-1\right)}\eta_2(w)dw \leq 0.002866,$$

we see that (6.34) implies (6.33). □

6.3. **Sums and decay for $\eta_+(t)$.** We will work with

$$\eta(t) = \eta_+(t) = h_H(t)\eta_\heartsuit(t) = h_H(t)e^{-t^2/2}, \tag{6.35}$$

where h_H is as in (4.10). Due to the sharp truncation in the Mellin transform Mh_H (see §4.2) and the pole of $M\eta_\heartsuit(s)$ at $s=0$, the Mellin transform $M\eta_+(s)$ of $\eta_+(t)$ has unpleasant singularities at $s=\pm iH$. In consequence, we must use a different contour of integration from the one we used before. This will require us to rework our explicit formula (Lemma 6.1) somewhat. We will need to assume that the non-trivial zeros of $L(s,\chi)$ lie on the critical line up to a height T_0; we would have needed to make the same assumption later anyhow.

[9]By rigorous integration from 1/4 to 1/2 and from 1/2 to 1 using VNODE-LP [Ned06].

Lemma 6.7. *Let $\eta : \mathbb{R}_0^+ \to \mathbb{R}$ be in C^1. Let $x \geq 1$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$.*

Let $H \geq 3/2$, $T_0 > H + 1$. Assume that all non-trivial zeros ρ of $L(s, \chi)$ with $|\Im(\rho)| \leq T_0$ satisfy $\Re(s) = 1/2$.

Write $G_\delta(s)$ for the Mellin transform of $\eta(t)e(\delta t)$. Assume that G_δ is holomorphic on $\{s : 1/5 \leq \Re(s) \leq 1 + \epsilon\} \cup \{s : -1/2 \leq \Re(s) \leq 1/5, |\Im(s)| \geq T_0 - 1\}$.

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \Lambda(n)\chi(n)e\left(\frac{\delta}{x}n\right)\eta(n/x) &= I_{q=1} \cdot \widehat{\eta}(-\delta)x - \sum_\rho G_\delta(\rho)x^\rho \\ &+ O^*((7.91 \log q + 82.7) \cdot (2\pi|\delta||\eta(t)t^{7/10}|_2 + |\eta'(t)t^{7/10}|_2)) \cdot x^{1/5} \\ &+ O^*\left(\frac{7 \log H_-}{10\pi} + \frac{7 \log q}{2\pi} + 11.04\right) \cdot \max_{\sigma \in [-\frac{1}{2}, \frac{1}{5}]} |G_\delta(\sigma + iH_-)|x^\sigma \\ &+ O^*\left(\frac{\log q + \log H_- + 6.71}{\sqrt{\pi H_-}} \cdot (2\pi|\delta||\eta|_2 + |\eta'|_2)\right) \cdot x^{-1/2},\end{aligned} \tag{6.36}$$

where

$$I_{q=1} = \begin{cases} 1 & \text{if } q = 1, \\ 0 & \text{if } q \neq 1, \end{cases} \qquad H_- = T_0 - 1$$

and the norms $|\eta|_2$, $|\eta'|_2$ are taken with respect to the usual measure dt. The sum $\sum_\rho$ is a sum over all non-trivial zeros ρ of $L(s, \chi)$.

Proof. We start just as in the proof of Lem. 6.1, except we shift the integral only up to $\Re(s) = 1/5$ in the central interval, and push it up to $\Re(s) = -1/2$ only in the tails:

$$\begin{aligned}\sum_{n=1}^{\infty} \Lambda(n)\chi(n)e(\delta n/x)\eta(n/x) &= I_{q=1}\widehat{\eta}(-\delta)x - \sum_\rho G_\delta(\rho)x^\rho \\ &\quad - \frac{1}{2\pi i}\int_C \frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s)x^s ds,\end{aligned} \tag{6.37}$$

where $I_{1,\chi} = 1$ if $q > 1$ and $\chi(-1) = 1$ and $I_{1,\chi} = 0$ otherwise, and C is the contour consisting of

(1) a segment $C_1 = [1/5 - iH_-, 1/5 + iH_-]$, where $H_- = T_0 - 1 > H$,
(2) the union C_2 of two horizontal segments:

$$C_2 = \left[-\frac{1}{2} - iH_-, \frac{1}{5} - iH_-\right] \cup \left[-\frac{1}{2} + iH_-, \frac{1}{5} + iH_-\right],$$

(3) the union C_3 of two rays

$$C_3 = \left[-\frac{1}{2} - i\infty, -\frac{1}{2} - iH_-\right] \cup \left[-\frac{1}{2} + iH_-, -\frac{1}{2} + i\infty\right].$$

We can still use (6.3) to estimate $L'(s,\chi)/L(s,\chi)$ (given $L'(1-s,\chi)/L(1-s,\chi)$). We estimate $\psi(z)$ for $\Re(z) \geq 4/5$ much as in (6.5): since $|t^2 + z^2| \geq 16/25$ for t

real,

$$\begin{aligned}\psi(z) &= \log z - \frac{1}{2z} + 2\cdot O^*\left(\int_0^\infty \frac{tdt}{\frac{16}{25}(e^{2\pi t}-1)}\right)\\ &= \log z - \frac{1}{2z} + \frac{50}{16}\cdot O^*\left(\frac{1}{(2\pi)^2}\Gamma(2)\zeta(2)\right)\\ &= \log z - \frac{1}{2z} + O^*\left(\frac{25}{192}\right) = \log z + O^*\left(\frac{145}{192}\right)\end{aligned}$$

For s with $\Re(s) = 1/5$,

$$\begin{aligned}\left|\cot\frac{\pi s}{2}\right| &= \left|\frac{e^{\frac{\pi}{10}i-\frac{\pi}{2}\tau}+e^{-\frac{\pi}{10}i+\frac{\pi}{2}\tau}}{e^{\frac{\pi}{10}i-\frac{\pi}{2}\tau}-e^{-\frac{\pi}{10}i+\frac{\pi}{2}\tau}}\right| = \left|-1+2\frac{e^{\frac{\pi}{10}i-\frac{\pi}{2}\tau}}{e^{\frac{\pi}{10}i-\frac{\pi}{2}\tau}-e^{-\frac{\pi}{10}i+\frac{\pi}{2}\tau}}\right|\\ &\leq \left|\frac{e^{\frac{\pi}{10}i}+e^{-\frac{\pi}{10}i}}{e^{\frac{\pi}{10}i}-e^{-\frac{\pi}{10}i}}\right| = \left|\cot\left(\frac{\pi}{10}\right)\right| \leq 3.07769.\end{aligned}$$

(The inequality is clear for $\tau \geq 0$; the case $\tau < 0$ follows by symmetry.) Similarly,

$$\left|\cot\frac{\pi(s+1)}{2}\right| = \left|\cot\left(\frac{3\pi}{5}\right)\right| \leq 0.32492.$$

For $\Re(s)$ arbitrary and $|\tau| = |\Im(s)| \geq H_-$,

$$\left|\cot\frac{\pi s}{2}\right| \leq \left|\frac{e^{\frac{\pi}{2}H_-}+e^{-\frac{\pi}{2}H_-}}{e^{\frac{\pi}{2}H_-}-e^{-\frac{\pi}{2}H_-}}\right| = \left|\coth\frac{\pi H_-}{2}\right|.$$

We now must estimate $L'(s,\chi)/L(s,\chi)$, where $\Re(s) \geq 4/5$ and $|\Im(s)| \leq H_-$. By a lemma of Landau's (see, e.g., [MV07, Lemma 6.3], where the constants are easily made explicit) based on the Borel-Carathéodory Lemma (as in [MV07, Lemma 6.2]), any function f analytic and zero-free on a disc $C_{s_0,R} = \{s : |s-s_0| \leq R\}$ of radius $R > 0$ around s_0 satisfies

$$\frac{f'(s)}{f(s)} = O^*\left(\frac{2R\log M/|f(s_0)|}{(R-r)^2}\right)$$

for all s with $|s-s_0| \leq r$, where $0 < r < R$ and M is the maximum of $|f(z)|$ on $C_{s_0,R}$. We set $s_0 = 3/2 + i\tau$ (where $\tau = \Im(s)$), $r = 1/2 + 1/5 = 7/10$, and let $R \to 1^-$, using the assumption that $L(s,\chi)$ has no non-trivial zeroes off the critical line with imaginary part $\leq H_- + 1 = T_0$.

We obtain

$$\frac{L'(s,\chi)}{L(s,\chi)} = O^*\left(\frac{200}{9}\log\frac{\max_{s\in C_{s_0,1}}|L(s,\chi)|}{|L(s_0,\chi)|}\right). \tag{6.38}$$

Clearly,

$$|L(s_0,\chi)| \geq \prod_p (1+p^{-3/2})^{-1} = \prod_p \frac{(1-p^{-3})^{-1}}{(1-p^{-3/2})^{-1}} = \frac{\zeta(3)}{\zeta(3/2)} \geq 0.46013.$$

By partial summation, for $s = \sigma + it$ with $\sigma \geq 1/2$ and any $N \in \mathbb{Z}^+$,

$$\begin{aligned} L(s,\chi) &= \sum_{n\leq N} \chi(m) n^{-s} - \left(\sum_{m\leq N} \chi(m)\right)(N+1)^{-s} \\ &\quad + \sum_{n\geq N+1} \left(\sum_{m\leq n} \chi(m)\right)(n^{-s} - (n+1)^{-s+1}) \\ &= O^*\left(\frac{N^{1-1/2}}{1-1/2} + N^{1-\sigma} + M(q)N^{-\sigma}\right) = O^*\left(3\sqrt{N} + M(q)/\sqrt{N}\right), \end{aligned}$$

where $M(q) = \max_n \left|\sum_{m\leq n} \chi(m)\right|$. We set $N = M(q)/3$, and obtain

$$|L(s,\chi)| \leq 2M(q)N^{-1/2} = 2\sqrt{3}\sqrt{M(q)}.$$

We can afford to use the trivial bound $M(q) \leq q$. We conclude that

$$\frac{L'(s,\chi)}{L(s,\chi)} = O^*\left(8 \log \frac{2\sqrt{3}\sqrt{q}}{\zeta(3)/\zeta(3/2)}\right) = O^*\left(\frac{100}{9}\log q + 44.8596\right).$$

Therefore, by (6.3) and (6.4),

$$\begin{aligned} \frac{L'(s,\chi)}{L(s,\chi)} &= \log\frac{\pi}{q} - \log(1-s) + O^*\left(\frac{145}{192}\right) \\ &\quad + \log 2 + O^*\left(\frac{\pi}{2}\cot\frac{\pi}{10}\right) + O^*\left(4\log q + 44.8596\right) \\ &= -\log(1-s) + O^*(5\log q + 52.2871) \end{aligned} \tag{6.39}$$

for $\Re(s) = 1/5$, $|\Im(s)| \leq H_-$,

$$\begin{aligned} \frac{L'(s,\chi)}{L(s,\chi)} &= \log\frac{\pi}{q} - \log(1-s) + O^*\left(\frac{145}{192}\right) \\ &\quad + \log 2 + O^*\left(\frac{\pi}{2}\coth\frac{\pi H_-}{2}\right) + O^*\left(4\log q + 44.8596\right) \\ &= -\log(1-s) + O^*(5\log q + 49.1654). \end{aligned} \tag{6.40}$$

for $\Re(s) \in [-1/2, 1/5]$, $1 \leq |\Im(s)| \leq H_-$, and (as in (6.7))

$$\frac{L'(s,\chi)}{L(s,\chi)} \leq -\log(1-s) + O^*(\log q + 5.2844). \tag{6.41}$$

Let $1 \leq j \leq 3$. By Cauchy-Schwarz,

$$\left|\frac{1}{2\pi}\int_{C_j} \frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) ds\right|$$

is at most

$$\sqrt{\frac{1}{2\pi}\int_{C_j} \left|\frac{L'(s,\chi)}{L(s,\chi)}\frac{1}{s}\right|^2 |ds|} \cdot \sqrt{\frac{1}{2\pi}\int_{C_j} |G_\delta(s)s|^2 |ds|}.$$

By (6.39) and (6.40),

$$\sqrt{\int_{C_j} \left|\frac{L'(s,\chi)}{L(s,\chi)}\right|^2 \frac{|ds|}{|s|^2}} \leq \sqrt{\int_{C_j} \left|\frac{5\log q}{s}\right|^2 |ds|} + \sqrt{\int_{C_j} \left|\frac{|\log|1-s|| + c_{2,j}}{s}\right|^2 |ds|},$$

where $c_{1,1} = 5$, $c_{1,2} = 5$, $c_{1,3} = 1$, $c_{2,1} = 52.2871$, $c_{2,2} = 49.1654$, $c_{2,3} = 5.2844$.

For $j = 1$,

$$\int_{C_1} \left|\frac{5\log q}{s}\right|^2 |ds| = (5\log q)^2 \cdot 10\tan^{-1} 5H_- \leq 125\pi(\log q)^2,$$

$$\int_{C_1} \left|\frac{|\log|1-s|| + c_{2,j}}{s}\right|^2 |ds| \leq \int_{-H_-}^{H_-} \frac{\left|\frac{1}{2}\log\left(\tau^2+\frac{16}{25}\right) + c_{2,j}\right|^2}{\frac{1}{25}+\tau^2} d\tau \leq 42949.3,$$

where we have computed the last integral numerically. As before, $G_\delta(s)s$ is the Mellin transform of (6.9). Hence

$$\begin{aligned}\sqrt{\frac{1}{2\pi}\int_{C_1} |G_\delta(s)s|^2 |ds|} &\leq \sqrt{\frac{1}{2\pi}\int_{\frac{1}{5}-i\infty}^{\frac{1}{5}+i\infty} |G_\delta(s)s|^2 |ds|} \\ &\leq \sqrt{\int_0^\infty \left|-2\pi i\delta te(\delta t)\eta(t) - te(\delta t)\eta'(t)\right|^2 t^{-3/5} dt} \\ &\leq 2\pi|\delta||\eta(t)t^{7/10}|_2 + |\eta'(t)t^{7/10}|_2.\end{aligned}$$

Hence,

$$\left|\frac{1}{2\pi}\int_{C_1} \frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) ds\right|$$

is at most

$$(7.9057\log q + 82.678) \cdot (2\pi|\delta||\eta(t)t^{7/10}|_2 + |\eta'(t)t^{7/10}|_2).$$

For $j = 3$,

$$\int_{C_3} \left|\frac{\log q}{s}\right|^2 |ds| = 2(\log q)^2 \int_{-1/2+iH_-}^{\infty} \frac{|ds|}{|s|^2} < \frac{2(\log q)^2}{H_-},$$

$$\begin{aligned}\int_{C_3} \left|\frac{|\log|1-s|| + c_{2,j}}{s}\right|^2 |ds| &\leq 2\int_{H_-}^{\infty} \frac{\left|\frac{1}{2}\log\left(\tau^2+\frac{9}{4}\right) + c_{2,j}\right|^2}{\frac{1}{4}+\tau^2} d\tau \\ &\leq 2\int_{H_-}^{\infty} \frac{\left|\log\tau + \frac{\log 2}{2} + c_{2,j}\right|^2}{\tau^2} d\tau \\ &\leq \frac{2(\log H_- + 6.71)^2}{H_-}.\end{aligned}$$

provided that $H_- \geq 3/2$. Now, as in (6.10),

$$\sqrt{\frac{1}{2\pi}\int_{C_3} |G_\delta(s)s|^2 |ds|} \leq 2\pi|\delta||\eta|_2 + |\eta'|_2.$$

Hence,

$$\left|\frac{1}{2\pi}\int_{C_1} \frac{L'(s,\chi)}{L(s,\chi)} G_\delta(s) ds\right| \leq \frac{\log q + \log H_- + 6.71}{\sqrt{\pi H_-}} \cdot (2\pi|\delta||\eta|_2 + |\eta'|_2).$$

Lastly, for $j = 2$,

$$\begin{aligned}\frac{1}{2\pi}\int_{C_2} \left|\frac{L'(s,\chi)}{L(s,\chi)}\right| |ds| &\leq \frac{\frac{1}{2}+\frac{1}{5}}{\pi} \cdot (\log|1-s| + O^*(5\log q + c_{2,2})) \\ &\leq \frac{7(\log H_- + 5\log q + 49.51198)}{10\pi}.\end{aligned}$$

□

Lemma 6.8. *Let $\eta = \eta_+$ be as in (6.35) for some $H \geq 5$. Let $x \in \mathbb{R}^+$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$. Assume that all non-trivial zeros ρ of $L(s,\chi)$ with $|\Im(\rho)| \leq T_0$ satisfy $\Re(s) = 1/2$, where $T_0 \geq H + \max(4\pi^2|\delta|, H/2, 100)$.*

Write $G_\delta(s)$ for the Mellin transform of $\eta(t)e(\delta t)$. Then

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |G_\delta(\rho)| \leq \sqrt{H} \log \frac{qT_0}{2\pi} \cdot \left(3.44 e^{-0.1598(T_0-H)} + 0.63|\delta| e^{-0.1065\frac{(T_0-H)^2}{(\pi\delta)^2}}\right).$$

Proof. Clearly,

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |G_\delta(\rho)| = \sum_{\substack{\rho \text{ non-trivial} \\ \Im(\rho)>T_0}} \left(|G_\delta(\rho)| + |G_\delta(1-\rho)|\right).$$

Let F_δ be as in (6.19). Then, since $\eta_+(t)e(\delta t) = h_H(t)e^{-t^2/2}e(\delta t)$, where h_H is as in (4.10), we see by (2.3) that

$$G_\delta(s) = \frac{1}{2\pi}\int_{-H}^{H} Mh(ir)F_\delta(s-ir)dr,$$

where F_δ is as in (6.19), and so, since $|Mh(ir)| = |Mh(-ir)|$,

$$|G_\delta(\rho)|+|G_\delta(1-\rho)| \leq \frac{1}{2\pi}\int_{-H}^{H} |Mh(ir)|(|F_\delta(\rho-ir)|+|F_\delta(1-(\rho-ir))|)dr. \tag{6.42}$$

We now proceed much as in the proof of Lem. 6.4. By Prop. 5.1, given $s = \rho + i\tau$, $\sigma \in [0,1]$, $\tau \geq 4\pi^2 \cdot \max(1, |\delta|)$,

$$|F_\delta(\rho)| + |F_\delta(1-\rho)|$$

is at most

$$C_{0,\tau,\ell} \cdot e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right)\cdot\tau} + C_{1,\tau} \cdot e^{-0.4798\tau} + C_{2,\tau} \cdot e^{-\min\left(\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2, \frac{25}{32}|\tau|\right)} + C_\tau' e^{-\frac{\pi}{4}|\tau|},$$

where $E(\rho)$ is as in (5.66),

$$C_{0,\tau,\ell} \leq 2\cdot(1+7.83^{1-\sigma})\left(\frac{3/2}{2\pi}\right)^{1-\sigma} \leq 4.217, \quad C_{1,\tau} = \left(1 + \frac{1+\sqrt{2}}{\tau}\right)\tau^{\frac{1}{2}},$$

$$C_{2,\tau} \leq \min\left(\frac{1}{4}\left(\frac{|\tau|}{\pi|\delta|}\right)^2, \frac{25}{16}|\tau|\right)^{-1}, \qquad C_\tau' \leq \frac{e^{\pi/2}\tau^{1/2}}{2}\left(1 + \frac{2\pi^{3/2}|\delta|}{\sqrt{|\tau|}}\right).$$

For $\tau \geq T_0 - H \geq 100$,

$$\begin{aligned}&\left(1 + \frac{1+\sqrt{2}}{\tau}\right)\tau^{\frac{1}{2}} + \frac{e^{\pi/2}\tau^{1/2}}{2}\left(1+\frac{2\pi^{3/2}|\delta|}{\sqrt{|\tau|}}\right)\cdot e^{-\left(\frac{\pi}{4}-0.4798\right)|\tau|}\\ &\leq 1.025|\tau|^{1/2} + 1.5\cdot 10^{-12}\cdot|\delta| \leq 0.033|\tau|\end{aligned}$$

and so

$$C_{1,\tau}\cdot e^{-0.4798|\tau|} + C_\tau' e^{-\frac{\pi}{4}|\tau|} \leq 0.033|\tau|e^{-0.4798|\tau|}.$$

It is clear that this is increasing for $|\tau| \geq 100$.

We bound $E(\rho)$ as in (6.21). Inequalities (6.22) and (6.23) still hold. We also see that, for $|\tau| \geq T_0 - H \geq \max(4\pi^2|\delta|, 100)$,

$$0.055\cdot C_{2,\tau} \leq 0.055\min(4\pi^2, 2500/16)^{-1} \leq 0.0014.$$

Thus,

$$C_{0,\tau,\ell}\cdot e^{-E\left(\frac{|\tau|}{(\pi\delta)^2}\right)\cdot\tau}+C_{2,\tau}\cdot e^{-\min\left(\frac{1}{8}\left(\frac{\tau}{\pi\delta}\right)^2,\frac{25}{32}|\tau|\right)}$$
$$\leq 4.22e^{-0.1598\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right)},$$

and so $|F_\delta(\rho)|+|F_\delta(1-\rho)|\leq g(\tau)$, where

$$g(\tau)=4.22e^{-0.1598\min\left(\frac{2}{3}\left(\frac{\tau}{\pi\delta}\right)^2,|\tau|\right)}+0.033|\tau|e^{-0.4798|\tau|}$$

is decreasing for $\tau\geq T_0-H$. Recall that, by (4.24),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|Mh_H(ir)|dr\leq 1.99301\sqrt{\frac{H}{2\pi}}.$$

Therefore, using (6.42), we conclude that

$$|G_\delta(\rho)|+|G_\delta(1-\rho)|\leq f(\tau),$$

for $\rho=\sigma+i\tau$, $\tau>0$, where

$$f(\tau)=0.7951\sqrt{H}\cdot g(\tau-H)$$

is decreasing for $\tau\geq T_0$ (because $g(\tau)$ is decreasing for $\tau\geq T_0-H$).

We apply Lemma 6.2, and get that

$$\sum_{\substack{\rho \text{ non-trivial}\\ |\Im(\rho)|>T_0}}|G_\delta(\rho)|\leq\int_{T_0}^{\infty}f(T)\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)dT$$
$$=0.7951\sqrt{H}\cdot\int_{T_0}^{\infty}g(T-H)\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)dT.$$

We continue as in the proof of Lemma 6.4, only the integrals are somewhat simpler this time. For any $c>0$,

$$\begin{aligned}\int_{T_0}^{\infty}&e^{-c(T-H)}\left(\frac{1}{2\pi}\log\frac{qT}{2\pi}+\frac{1}{4T}\right)dT\\&\leq\left(\frac{1}{2\pi c}\log\frac{qT_0}{2\pi}+\left(\frac{1}{2\pi c^2}+\frac{1}{4c}\right)\frac{1}{T_0}\right)e^{-c(T_0-H)}.\end{aligned}\tag{6.43}$$

At the same time,

$$\int_{T_0}^{\infty}e^{-\frac{2}{3}c\left(\frac{T-H}{\pi\delta}\right)^2}\left(\frac{1}{2\pi}\log\frac{qT_0}{2\pi}+\frac{1}{4T}\right)dT$$
$$=\int_{\frac{T_0}{\pi|\delta|}}^{\infty}e^{-\frac{2}{3}c\left(t-\frac{H}{\pi\delta}\right)^2}\left(\frac{\pi|\delta|}{2\pi}\log\frac{q|\delta|t}{2}+\frac{1}{4t}\right)dt\leq\frac{\left(\frac{|\delta|}{2}\log\frac{qT_0}{2\pi}+\frac{\pi|\delta|}{4T_0}\right)}{\frac{4}{3}c\frac{T_0-H}{\pi|\delta|}}e^{-\frac{2}{3}c\left(\frac{T_0-H}{\pi\delta}\right)^2},$$

since $T_0\geq 4\pi^2e>2\pi e$. For $c_1>0$, since

$$\left(\frac{T+1/c_1}{2\pi c_1}\log\frac{qT}{2\pi}+\frac{1+\frac{1}{c_1T_0}}{2\pi c_1^2}+\frac{1}{4c_1}\right)e^{-c_1(T-H)},$$

we see that

$$\begin{aligned}
&\int_{T_0}^{\infty} T e^{-c_1(T-H)} \left(\frac{1}{2\pi}\log\frac{qT}{2\pi} + \frac{1}{4T}\right) dT \\
&\leq \left(\frac{1+\frac{1}{cT_0}}{2\pi c}\left(T_0 \log\frac{qT_0}{2\pi} + \frac{1}{c}\right) + \frac{1}{4c}\right) e^{-c_1(T_0-H)} \\
&\leq \left(\frac{1+\frac{1}{c_1T_0}}{2\pi c_1}\left(T_0 \log\frac{qT_0}{2\pi} + \frac{1}{c_1}\right) + \frac{1}{4c_1}\right) e^{-c_1(T_0-H)}.
\end{aligned} \tag{6.44}$$

We set $c = 0.1598$, $c_1 = 0.4798$. Since $T_0 \geq \max(100 + H, 3H/2)$, the ratio of the right side of (6.44) to the right side of (6.43) is at most

$$\begin{aligned}
&\max\left(\frac{c}{c_1}, \frac{c^2}{c_1^2}\right) \cdot \left(T_0 + \frac{1}{c_1}\right) e^{-(c_1-c_0)(T_0-H)} \\
&\qquad \leq \frac{c}{c_1} \cdot \left(3(T_0 - H) + \frac{1}{c_1}\right) e^{-(c_1-c_0)(T_0-H)} \leq 1.3 \cdot 10^{-12}.
\end{aligned}$$

We also see that

$$0.7951 \cdot 4.22 \left(\frac{1}{2\pi c} + \frac{1}{T_0}\frac{\frac{1}{2\pi c^2} + \frac{1}{4c}}{\log\frac{qT_0}{2\pi}}\right) \leq 3.4303$$

and, since $T_0 - H \geq 4\pi^2|\delta|$,

$$\begin{aligned}
0.7951 \cdot 4.22 \cdot \frac{\left(\frac{1}{2}\log\frac{qT_0}{2\pi} + \frac{\pi}{4T_0}\right)}{\frac{4}{3}c\frac{T_0-H}{\pi|\delta|}} &\leq 3.35533 \cdot \frac{\frac{1}{2} + \frac{\pi}{4T_0\log\frac{T_0}{2\pi}}}{\frac{4}{3}c \cdot 4\pi} \log\frac{qT_0}{2\pi} \\
&\leq 0.62992 \log\frac{qT_0}{2\pi}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&0.7951 \int_{T_0}^{\infty} g(T-H) \left(\frac{1}{2\pi}\log\frac{qT}{2\pi} + \frac{1}{4T}\right) dT \\
&\qquad \leq \log\frac{qT_0}{2\pi} \cdot \left(3.44 e^{-0.1598(T_0-H)} + 0.63|\delta| e^{-0.1065\frac{(T_0-H)^2}{(\pi\delta)^2}}\right).
\end{aligned}$$

□

Proposition 6.9. *Let $\eta = \eta_+$ be as in (6.35) for some $H \geq 50$. Let $x \geq 10^3$, $\delta \in \mathbb{R}$. Let χ be a primitive character mod q, $q \geq 1$. Assume that all non-trivial zeros ρ of $L(s,\chi)$ with $|\Im(\rho)| \leq T_0$ lie on the critical line, where $T_0 \geq H + \max(4\pi^2|\delta|, H/2, 100)$.*

Then

$$\sum_{n=1}^{\infty} \Lambda(n)\chi(n) e\left(\frac{\delta}{x}n\right) \eta_+(n/x) = \begin{cases} \widehat{\eta_+}(-\delta)x + O^*\left(\mathrm{err}_{\eta_+,\chi}(\delta,x)\right) \cdot x & \text{if } q = 1, \\ O^*\left(\mathrm{err}_{\eta_+,\chi}(\delta,x)\right) \cdot x & \text{if } q > 1, \end{cases} \tag{6.45}$$

where

(6.46)

$$\mathrm{err}_{\eta_+,\chi}(\delta,x) = \sqrt{H}\log\frac{qT_0}{2\pi}\cdot\left(3.44e^{-0.1598(T_0-H)} + 0.63|\delta|e^{-0.1065\frac{(T_0-H)^2}{(\pi\delta)^2}}\right)$$
$$+O^*(((0.641+1.11\sqrt{H})\log qT_0 + (1.5+19.1\sqrt{H}))\sqrt{T_0} + 1.65\log q + 44)x^{-\frac{1}{2}}$$
$$+O^*\left(|\delta|(40.2\log q + 420) + \sqrt{H}(0.015\log T_0 + 15.6\log q + 163)\right)x^{-\frac{4}{5}}.$$

Proof. We apply Lemmas 6.7, Lemma 6.3 and Lemma 6.8. We bound the norms involving η_+ using the estimates in §4.2.2. The error terms in (6.36) total at most

$$(6.47)\qquad \begin{aligned}&((7.91\log q + 82.7)(5.074|\delta| + 1.953\sqrt{H}))\cdot x^{1/5}\\ &+(0.23\log T + 1.12\log q + 11.04)x^{1/5}\cdot \max_{\sigma\in[-\frac{1}{2},\frac{1}{5}]}|G_\delta(\sigma+i(T-1))|\\ &+(0.05\log q + 0.542)(4.027|\delta| + 0.876\sqrt{H})x^{-1/2}\end{aligned}$$

Since $x\geq 10^3$, the last line of (6.47) is easily absorbed into the first line of (6.47) (by a change in the last significant digits). Much as in the proof of Lem. 6.8, we bound

$$\begin{aligned}|G_\delta(\sigma+i(T-1))| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}|Mh_H(ir)|dr\cdot\max_{\tau\geq T-1-H}|F_\delta(\sigma+i\tau)|\\ &\leq 0.3172\sqrt{H}\max_{\tau\geq T-1-H}|F_\delta(\sigma+i\tau)|,\end{aligned}$$

where F_δ is as in (6.19). We can obtain an easy bound for F_δ applicable for σ arbitrary as follows:

$$-sF_\delta(s) = M\left(t\frac{d}{dt}\left(e^{-\frac{t^2}{2}}e(\delta t)\right)\right)(s) = M\left((-t^2e(\delta t) + 2\pi i\delta te(\delta t))\eta_\heartsuit(t)\right)(s),$$

and so

$$\begin{aligned}|sF_\delta(s)| &\leq \int_0^\infty |t+2\pi\delta i|\eta_\heartsuit(t)t^{\sigma+1}\frac{dt}{t} \leq |\eta_\heartsuit(t)t^{\sigma+1}|_1 + 2\pi|\delta||\eta_\heartsuit(t)t^\sigma|_1\\ &= 2^{\sigma/2}\Gamma\left(\frac{\sigma}{2}+1\right) + 2^{\frac{\sigma+1}{2}}\pi|\delta|\cdot\Gamma\left(\frac{\sigma+1}{2}\right).\end{aligned}$$

Since $\sigma\in[-1/2,1/5]$ and $x\geq 10^3$, a quick verification gives that $x^\sigma 2^{\sigma/2}\Gamma(\sigma/2+1)$ and $x^\sigma 2^{(\sigma+1)/2}\pi|\delta|\cdot\Gamma((\sigma+1)/2)$ are maximal when $\sigma = 1/5$. Hence

$$\begin{aligned}|F_\delta(s)|x^\sigma &\leq \frac{1.01964 + 7.09119|\delta|}{|s|}x^{1/5} \leq \frac{1.01964+7.09119|\delta|}{\frac{99}{100}\max(4\pi^2|\delta|,100)}x^{1/5}\\ &\leq (0.0103 + 0.18144)x^{1/5} \leq 0.19174x^{1/5},\end{aligned}$$

and so

$$|G_\delta(\sigma+i(T-1))| \leq 0.3172\sqrt{H}\cdot 0.19174 \leq 0.061\sqrt{H}.$$

We conclude that (6.47) is at most

$$(6.48)\qquad \begin{aligned}&(5.075|\delta| + 1.954\sqrt{H})\cdot(7.91\log q + 82.7)\cdot x^{1/5}\\ &+0.061\sqrt{H}\cdot(0.23\log T + 1.12\log q + 11.04)\cdot x^{1/5}\\ &\leq (|\delta|(40.2\log q + 420) + \sqrt{H}(0.015\log T + 15.6\log q + 163))x^{1/5}.\end{aligned}$$

□

Finally, let us prove a simple result that will allow us to compute a key ℓ_2 norm.

Proposition 6.10. *Let $\eta = \eta_+$ be as in (6.35), $H \geq 50$. Let $x \geq 10^6$. Assume that all non-trivial zeros ρ of the Riemann zeta function $\zeta(s)$ with $|\Im(\rho)| \leq T_0$ lie on the critical line, where $T_0 \geq 2H + \max(H, 100)$.*

Then

$$\sum_{n=1}^{\infty} \Lambda(n)(\log n)\eta_+^2(n/x) = x \cdot \int_0^{\infty} \eta_+^2(t) \log xt\, dt + O^*(\mathrm{err}_{\ell_2,\eta_+}) \cdot x \log x, \tag{6.49}$$

where

$$\begin{aligned}\mathrm{err}_{\ell_2,\eta_+} = &\left(0.311 \frac{(\log T_0)^2}{\log x} + 0.224 \log T_0\right) H\sqrt{T_0} e^{-\pi(T_0-2H)/4} \\ &+ (6.2\sqrt{H} + 5.3)\sqrt{T_0} \log T_0 \cdot x^{-1/2} + 419.3\sqrt{H} x^{-4/5}.\end{aligned} \tag{6.50}$$

Proof. We will need to consider two smoothing functions, namely, $\eta_{+,0}(t) = \eta_+(t)^2$ and $\eta_{+,1} = \eta_+(t)^2 \log t$. Clearly,

$$\sum_{n=1}^{\infty} \Lambda(n)(\log n)\eta_+^2(n/x) = (\log x) \sum_{n=1}^{\infty} \Lambda(n)\eta_{+,0}(n/x) + \sum_{n=1}^{\infty} \Lambda(n)\eta_{+,1}(n/x).$$

Since $\eta_+(t) = h_H(t)e^{-t^2/2}$,

$$\eta_{+,0}(r) = h_H^2(t)e^{-t^2}, \qquad \eta_{+,1}(r) = h_H^2(t)(\log t)e^{-t^2}.$$

Let $\eta_{+,2} = (\log x)\eta_{+,0} + \eta_{+,1}$.

The Mellin transform of e^{-t^2} is $\Gamma(s/2)/2$; by (2.5), this implies that the Mellin transform of $(\log t)e^{-t^2}$ is $\Gamma'(s/2)/4$. Hence, by (2.3),

$$M\eta_{+,0}(s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} Mh_H^2(ir) \cdot F_x\left(\frac{s - ir}{2}\right) dr, \tag{6.51}$$

where

$$F_x(s) = (\log x)\Gamma(s) + \frac{1}{2}\Gamma'(s). \tag{6.52}$$

Moreover,

$$Mh_H^2(ir) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Mh_H(iu) Mh_H(i(r-u))du, \tag{6.53}$$

and so $Mh_H^2(ir)$ is supported on $[-2H, 2H]$. We also see that $|Mh_H^2(ir)|_1 \leq |Mh_H(ir)|_1^2/2\pi$. We know that $|Mh_H(ir)|_1^2/2\pi \leq 1.99301^2 H$ by (4.24).

Hence

(6.54)

$$\begin{aligned}|M\eta_{+,0}(s)| &\leq \frac{1}{4\pi} \int_{-\infty}^{\infty} |M(h_H^2)(ir)| dr \cdot \max_{|r| \leq 2H} |F_x((s - ir)/2)| \\ &\leq \frac{1.99301^2}{4\pi} H \cdot \max_{|r|\leq 2H} |F_x((s-ir)/2)| \leq 0.3161 H \cdot \max_{|r| \leq 2H} |F_x((s - ir)/2)|.\end{aligned}$$

By [OLBC10, 5.6.9] (Stirling with explicit constants),

$$|\Gamma(s)| \leq \sqrt{2\pi} |s|^{\Re(s)-1/2} e^{-\pi|\Im(s)|/2} e^{1/6|z|}, \tag{6.55}$$

and so

$$|\Gamma(s)| \leq 2.511 \sqrt{|\Im(s)|} e^{-\pi|\Im(s)|/2} \tag{6.56}$$

for $s \in \mathbb{C}$ with $-1 \leq \Re(s) \leq 1$ and $|\Im(s)| \geq 99$. Moreover, by [OLBC10, 5.11.2] and the remarks at the beginning of [OLBC10, 5.11(ii)],

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O^*\left(\frac{1}{\cos^3 \theta/2}\right)$$

for $|\arg(s)| < \theta$ ($\theta \in (-\pi, \pi)$). Again, $s = \sigma + i\tau$ with $-1 \leq \sigma \leq 1$ and $|\tau| \geq 99$, this gives us

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log|\tau| + O^*(3.02).$$

Hence, under the same conditions on s,

$$\begin{aligned} |F_x(s)| &\leq ((\log x) + \frac{1}{2}\log|\tau| + 1.51)\Gamma(s) \\ &\leq 2.511((\log x) + \frac{1}{2}\log|\tau| + 1.51)\sqrt{|\tau|}e^{-\pi|\tau|/2}. \end{aligned} \tag{6.57}$$

Thus, by (6.54),

$$|M\eta_{+,2}(\rho)| \leq 0.7938H((\log x) + \frac{1}{2}\log|\tau| + 1.17)\sqrt{\frac{|\tau|}{2} - H}e^{-\pi(|\tau|-2H)/4} \tag{6.58}$$

for $\rho = \sigma + i\tau$ with $|\tau| \geq T_0 - 1 \geq 2H + 99$ and $-1 \leq \sigma \leq 1$.

We now apply Lemma 6.7 with $\eta = \eta_{+,2}$, $\delta = 0$ and χ trivial. We obtain that

$$\begin{aligned} &\sum_{n=1}^{\infty} \Lambda(n)\eta_{+,2}(n/x) \\ &= \left(\int_0^\infty \eta_{+,2}(t)dt\right)\cdot x - \sum_\rho M\eta_{+,2}(\rho)x^\rho + O^*\left(82.7|\eta'_{+,2}(t)t^{7/10}|_2\right)x^{1/5} \\ &+ O^*\left(\frac{7\log T_0}{10\pi} + 11.04\right)\cdot \max_{\sigma\in[-\frac{1}{2},\frac{1}{5}]} |M\eta_{+,2}(\sigma + i(T_0 - 1))|\cdot x^\sigma \\ &+ O^*\left(\frac{\log T_0 + 6.71}{\sqrt{\pi(T_0-1)}}|\eta'_{+,2}|_2\right)\cdot x^{-1/2} \end{aligned} \tag{6.59}$$

Since $\eta_{+,2} = (\log xt)\eta_+^2$, we can bound

$$\begin{aligned} |\eta'_{+,2}(t)t^{7/10}|_2 &\leq |2\eta_+\eta'_+(\log xt)t^{7/10}|_2 + |\eta_+^2 t^{-3/10}|_2 \\ &\leq 2(|\eta_+|_\infty \log x + |\eta_+(t)\log t|_\infty)|\eta'_+ t^{7/10}|_2 + |\eta_+|_\infty|\eta_+ t^{-3/10}|_2 \\ &\leq 2(1.26499\log x + 0.43088)\cdot 1.95201\sqrt{H} + 1.26499\cdot 0.66241 \\ &\leq 4.93855\sqrt{H}\log x + 1.68217\sqrt{H} + 0.83795 \end{aligned}$$

by (4.31), (4.32), (4.22) and (4.19), using the assumption $H \geq 50$. Similarly,

$$\begin{aligned} |\eta'_{+,2}|_2 &\leq 2(|\eta_+|_\infty \log x + |\eta_+(t)\log t|_\infty)|\eta'_+|_2 + |\eta_+|_\infty|\eta_+|_2 \\ &\leq 2(1.26499\log x + 0.43088)\cdot 0.87531\sqrt{H} + 1.26499\cdot 0.80075 \\ &\leq 2.21452\sqrt{H}\log x + 0.75431\sqrt{H} + 1.01295 \end{aligned}$$

by (4.32), (4.21) and (4.19). Hence, since $H \geq 50$ and $x \geq 10^6$,

$$\begin{aligned}
&82.7|\eta'_{+,2}(t)t^{7/10}|_2 x^{1/5} + \frac{\log T_0 + 6.71}{\sqrt{\pi(T_0-1)}}|\eta'_{+,2}|_2 \cdot x^{-(1/2+1/5)}\\
&\quad\leq (408.5\sqrt{H}\log x + 139.2\sqrt{H} + 69.3)x^{1/5}\\
&\quad+ (1.07\sqrt{H}\log x + 0.363\sqrt{H} + 0.487)x^{-1/2} \leq 419.29\sqrt{H}x^{1/5}\log x.
\end{aligned}$$

We bound $M\eta_{+,2}$ by (6.58):

(6.60)

$$\begin{aligned}
&\left(\frac{7\log T_0}{10\pi} + 11.04\right) \cdot |M\eta_{+,2}(\sigma + i(T_0-1))|\\
&\leq (0.1769\log T_0 + 8.7628)(\log x + \frac{1}{2}\log T_0 + 1.17)H\sqrt{\frac{T_0}{2} - H}e^{-\pi(T_0-1-2H)/4}.
\end{aligned}$$

Since $T_0 \leq 3(T_0 - 2H)$, $H \leq T_0/3$ and $T_0 - 2H \geq 100$, this gives us that

$$\begin{aligned}
\left(\frac{7\log T_0}{10\pi} + 11.04\right) \cdot |M\eta_{+,2}(\sigma + i(T_0-1))| &\leq 3.54 \cdot 10^{-30}\log x + 1.542 \cdot 10^{-29}\\
&\leq 5 \cdot 10^{-30}\log x
\end{aligned}$$

for $-1 \leq \sigma \leq 1$ and $x \geq 10^6$. Thus, the error terms in (6.59) total at most

$$419.3\sqrt{H}x^{1/5}\log x.$$

It is time to bound the contribution of the zeros. The contribution of the zeros up to T_0 gets bounded by Lemma 6.3:

$$\begin{aligned}
\sum_{\substack{\rho \text{ non-trivial}\\ |\Im(\rho)| \leq T_0}} |M\eta_{+,2}(\rho)| &\leq (|\eta_{+,2}|_2 + |\eta_{+,2} \cdot \log|_2)\sqrt{T_0}\log T_0\\
&+ 17.21|\eta_{+,2} \cdot \log|_2\sqrt{T_0} + 34.5\left|\eta_{+,2}(t)/\sqrt{t}\right|_1
\end{aligned}$$

Since $\eta_{+,2} = (\log xt)\eta_+^2$, we bound the norms here as follows:

$$\begin{aligned}
|\eta_{+,2}|_2 &\leq (|\eta_+|_\infty \log x + |\eta_+ \cdot \log|_\infty)|\eta_+|_2\\
&\leq 1.014\log x + 0.346 \leq 1.04\log x,\\
|\eta_{+,2} \cdot \log|_2 &\leq (|\eta_+|_\infty \log x + |\eta_+ \cdot \log|_\infty)|\eta_+ \cdot \log|_2\\
&\leq (1.404\log x + 0.479)\sqrt{H} \leq 1.44\sqrt{H}\log x,\\
|\eta_{+,2}/\sqrt{t}|_1 &\leq (|\eta_+|_\infty \log x + |\eta_+ \cdot \log|_\infty)|\eta_+/\sqrt{t}|_1\\
&\leq 1.679\log x + 0.572 \leq 1.721\log x,
\end{aligned}$$

where we use the bounds $|\eta_+|_\infty \leq 1.265$, $|\eta_+ \cdot \log|_\infty \leq 0.431$, $|\eta_+|_2 \leq 0.8008$, $|\eta_+ \cdot \log|_2 \leq 1.1096\sqrt{H}$, $|\eta(t)/\sqrt{t}|_1 \leq 1.3267$ from (4.31), (4.32) and (4.17) (with $H \geq 50$) and (4.26). (We also use the assumption $x \geq 10^6$.) Since $T_0 \geq 200$, this means that

$$\sum_{\substack{\rho \text{ non-trivial}\\ |\Im(\rho)| \leq T_0}} |M\eta_{+,2}(\rho)| \leq (6.2\sqrt{H} + 5.3)\sqrt{T_0}\log T_0 \log x.$$

To bound the contribution of the zeros beyond T_0, we apply Lemma 6.2, and get that

$$\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |M\eta_{+,2}(\rho)| \leq \int_{T_0}^{\infty} f(T) \left(\frac{1}{2\pi}\log\frac{T}{2\pi} + \frac{1}{4T}\right) dT, \tag{6.61}$$

where

$$f(T) = 1.5876H \cdot ((\log x) + \frac{1}{2}\log|\tau| + 1.17)\sqrt{\frac{|\tau|}{2}} e^{-\pi(|\tau|-2H)/4}$$

(see (6.58)). Since $T \geq T_0 \geq 200$, we know that $((1/2\pi)\log(T/2\pi) + 1/4T) \geq 0.216\log T$. In general,

$$\int_{T_0}^{\infty} \sqrt{T}(\log T)^2 e^{-\pi T/4} dT \leq \frac{4}{\pi}\left(\sqrt{T_0}(\log T_0)^2 + \frac{2/\pi}{\sqrt{T_0}}(\log e^2 T_0)^2\right) e^{-\frac{\pi T_0}{4}},$$
$$\int_{T_0}^{\infty} \sqrt{T}(\log T) e^{-\pi T/4} dT \leq \frac{4}{\pi}\left(\sqrt{T_0}(\log T_0) + \frac{2/\pi}{\sqrt{T_0}}\log e^2 T_0\right) e^{-\frac{\pi T_0}{4}},$$
$$\int_{T_0}^{\infty} \sqrt{T} e^{-\pi T/4} dT = \frac{4}{\pi}\left(\sqrt{T_0} + \frac{2/\pi}{\sqrt{T_0}}\right) e^{-\frac{\pi T_0}{4}};$$

for $T_0 \geq 200$, the quantities on the right are at most $1.281 \cdot \sqrt{T_0}(\log T_0)^2 e^{-\pi T_0/4}$, $1.279\sqrt{T_0}(\log T_0)e^{-\pi T_0/4}$, and $1.278\sqrt{T_0}e^{-\pi T_0/4}$, respectively. Thus, (6.61) gives us that

$$\begin{aligned}\sum_{\substack{\rho \text{ non-trivial} \\ |\Im(\rho)|>T_0}} |M\eta_{+,2}(\rho)| &\leq 1.281 \cdot (0.216\log T_0) \cdot f(T_0) \\ &\leq (0.311(\log T_0)^2 + 0.224(\log x)(\log T_0)) H\sqrt{T_0} e^{-\pi(T_0-2H)/4}.\end{aligned}$$

□

6.4. **A verification of zeros and its consequences.** David Platt verified in his doctoral thesis [Pla11], that, for every primitive character χ of conductor $q \leq 10^5$, all the non-trivial zeroes of $L(s,\chi)$ with imaginary part $\leq 10^8/q$ lie on the critical line, i.e., have real part exactly 1/2. (We call this a *GRH verification up to* $10^8/q$.)

In work undertaken in coordination with the present project [Plab], Platt has extended this computations to

- all odd $q \leq 3 \cdot 10^5$, with $T_q = 10^8/q$,
- all even $q \leq 4 \cdot 10^5$, with $T_q = \max(10^8/q, 200 + 7.5 \cdot 10^7/q)$.

The method used was rigorous; its implementation uses interval arithmetic.

Let us see what this verification gives when used as an input to Cor. 6.6 and Prop. 6.9. Since we intend to apply these results to the estimation of (3.36), our main goal is to bound

$$E_{\eta,r,\delta_0} = \max_{\substack{\chi \bmod q \\ q \leq \gcd(q,2)\cdot r \\ |\delta| \leq \gcd(q,2)\delta_0 r/2q}} \sqrt{q} \cdot |\operatorname{err}_{\eta,\chi^*}(\delta, x)| \tag{6.62}$$

for $\eta = \eta_*$ and $\eta = \eta_+$. In the case of η_+, we will be able to assume $x \geq x_+ = 10^{29}$. In the case of η_*, we will prefer to work with a smaller x; thus, we make only the assumption $x \geq x_- = 10^{26}$. Since we will use Platt's input, we set $r = 150000$.

(Note that Platt's calculations really allow us to go up to $r = 200000$.) We also set $\delta_0 = 8$.

In general,

$$q \leq \gcd(q,2) \cdot r \leq 2r, \quad |\delta| \leq \frac{4r}{q/\gcd(q,2)}.$$

To work with Cor. 6.6, we set

$$T_0 = \frac{5 \cdot 10^7}{q/\gcd(q,2)}.$$

Thus

$$T_0 \geq \frac{5 \cdot 10^7}{150000} = \frac{1000}{3},$$

$$\frac{T_0}{\pi\delta} \geq \frac{5 \cdot 10^7}{4\pi r} = \frac{1000}{12\pi} = 26.525823\ldots$$

and so

$$3.5 \cdot e^{-0.1598 T_0} + 0.0074 \cdot e^{-0.1065 \cdot \frac{T_0^2}{(\pi\delta)^2}} \leq 2.575 \cdot 10^{-23}.$$

Since there are no primitive characters of modulus 2, $\delta\sqrt{q} \leq 4r$. Examining (6.33), we obtain

$$\begin{aligned}
&\sqrt{q} \cdot \mathrm{err}_{\eta_*,\chi} \leq \frac{10^8}{\sqrt{q}} \log \frac{10^8}{2\pi} \cdot 2.575 \cdot 10^{-23} \\
&+ \left(1.22\sqrt{10^8} \log 10^8 + 5.056\sqrt{10^8} + 1.423\sqrt{300000} \log 300000 + 38.19\sqrt{300000}\right) \\
&\cdot x_-^{-\frac{1}{2}+\epsilon} + (\log 300000 + 6.01) \cdot (0.89\sqrt{300000} + 2.7 \cdot 4 \cdot 300000) \cdot x_-^{-3/2} \\
&\leq 4.743 \cdot 10^{-14} + 3.0604 \cdot 10^{-8} + 6.035 \cdot 10^{-32} = 3.061 \cdot 10^{-8}.
\end{aligned}$$

To work with Prop. 6.9, we set

$$T_0 = H + \frac{3.75 \cdot 10^7}{q/\gcd(q,2)}, \qquad H = 200.$$

Thus

$$T_0 - H \geq \frac{3.75 \cdot 10^7}{150000} = 250,$$

$$\frac{T_0 - H}{\pi\delta} \geq \frac{3.75 \cdot 10^7}{4\pi r} = \frac{750}{12\pi} = 19.89436\ldots$$

and also

$$qT_0 \leq 2r \cdot H + 7.5 \cdot 10^7 \leq 1.35 \cdot 10^8.$$

Hence

$$3.44 \cdot \sqrt{2r} e^{-0.1598(T_0 - H)} + 0.63 \cdot 4r \cdot e^{-0.1065 \cdot \frac{(T_0-H)^2}{(\pi\delta)^2}} \leq 1.953 \cdot 10^{-13}.$$

Examining (6.46), we get

$$\begin{aligned}&\sqrt{q}\cdot \mathrm{err}_{\eta_+,\chi}(\delta,x) \leq \sqrt{200}\cdot\log\left(1.35\cdot 10^8\right)\cdot 1.953\cdot 10^{-13}\\&+\left((16.339\cdot\log\left(1.35\cdot 10^8\right)+271.62)\sqrt{1.35\cdot 10^8}+64.81\cdot\sqrt{2r}\right)\cdot x_+^{-\frac{1}{2}}\\&+\left(3708r+\sqrt{200r}\cdot 360.01\right)\cdot x_+^{-\frac{4}{5}}\\&\leq 5.1707\cdot 10^{-11}+2.1331\cdot 10^{-8}+3.5219\cdot 10^{-15}\leq 2.139\cdot 10^{-8}.\end{aligned}$$

We record our final conclusions: for E_{η,r,δ_0} defined as in (6.62) and $r = 150000$,

$$E_{\eta_*,r,8} \leq 3.061\cdot 10^{-8},\qquad E_{\eta_+,r,8}\leq 2.139\cdot 10^{-8}, \tag{6.63}$$

where we assume $x \geq x_- = 10^{26}$ when bounding $E_{\eta_*,r,8}$ and $x \geq x_+ = 10^{29}$ when bounding $E_{\eta_+,r,8}$.

Let us optimize things a little more carefully for the trivial character χ_T. We will make the stronger assumption $x \geq x_1 = 4.5\cdot 10^{29}$. We wish to bound $ET_{\eta_+,4r}$, where

$$ET_{\eta,s} = \max_{|\delta|\leq s} |\,\mathrm{err}_{\eta,\chi_T}(\delta,x)|.$$

We will go up to a height $T_0 = H + 600000\pi\cdot t$, where $H = 200$ and $t \geq 10$. Then

$$\frac{T_0-H}{\pi\delta}\geq\frac{200+600000\pi t}{4\pi r}\geq t.$$

Hence

$$3.44e^{-0.1598(T_0-H)}+0.63|\delta|e^{-0.1065\frac{(T_0-H)^2}{(\pi\delta)^2}}\leq 10^{-1300000}+156000e^{-0.1065t^2}.$$

Looking at (6.46), we get

$$\begin{aligned}ET_{\eta_+,4r} &\leq \sqrt{200}\log\frac{T_0}{2\pi}\cdot\left(10^{-1300000}+378000e^{-0.1065t^2}\right)\\&+O^*\left((16.339\log T_0+271.615)\sqrt{T_0}+44\right)\cdot x_1^{-1/2}\\&+O^*\left(2.52\cdot 10^8+\sqrt{200}(0.015\log T_0+163)\right)x_1^{-4/5}.\end{aligned}$$

We choose $t = 20$; this gives $T_0 \leq 3.77\cdot 10^7$, which is certainly within the checked range. We obtain

$$ET_{\eta_+,4r}\leq 5.122\cdot 10^{-9}. \tag{6.64}$$

for $r = 150000$ and $x \geq x_1 = 4.5\cdot 10^{-29}$.

Lastly, let us look at the sum estimated in (6.49). Here it will be enough to go up to just $T_0 = 3H = 600$. We make, again, the strong assumption $x \geq x_1 = 4.5\cdot 10^{29}$. We look at (6.50) and obtain

$$\begin{aligned}\mathrm{err}_{\ell_2,\eta_+} &\leq \left(0.311\frac{(\log 600)^2}{\log x_1}+0.224\log 600\right)\cdot 200\cdot\sqrt{600}e^{-50\pi}\\&+(6.2\sqrt{200}+5.3)\sqrt{600}\log 600\cdot x_1^{-1/2}+419.3\sqrt{200}x_1^{-4/5}\\&\leq 4.8\cdot 10^{-65}+2.17\cdot 10^{-11}+1.13\cdot 10^{-20}\leq 2.2\cdot 10^{-11}.\end{aligned} \tag{6.65}$$

It remains only to estimate the integral in (6.49). First of all,

$$\int_0^\infty \eta_+^2(t)\log xt\,dt = \int_0^\infty \eta_\circ^2(t)\log xt\,dt + 2\int_0^\infty (\eta_+(t)-\eta_\circ(t))\eta_\circ(t)\log xt\,dt + \int_0^\infty (\eta_+(t)-\eta_\circ(t))^2\log xt\,dt.$$

The main term will be given by

$$\int_0^\infty \eta_\circ^2(t)\log xt\,dt = \left(0.64020599736635 + O\left(10^{-14}\right)\right)\log x - 0.021094778698867 + O\left(10^{-15}\right),$$

where the integrals were computed rigorously using VNODE-LP [Ned06]. (The integral $\int_0^\infty \eta_\circ^2(t)dt$ can also be computed symbolically.) By Cauchy-Schwarz and the triangle inequality,

$$\begin{aligned}\int_0^\infty (\eta_+(t)-\eta_\circ(t))\eta_\circ(t)\log xt\,dt &\le |\eta_+-\eta_\circ|_2|\eta_\circ(t)\log xt|_2\\ &\le |\eta_+-\eta_\circ|_2(|\eta_\circ|_2\log x + |\eta_\circ\cdot\log|_2)\\ &\le \frac{547.56}{H^{7/2}}(0.80013\log x + 0.04574)\\ &\le 3.873\cdot 10^{-6}\cdot\log x + 2.214\cdot 10^{-7},\end{aligned}$$

where we are using (4.14) and evaluate $|\eta_\circ\cdot\log|_2$ rigorously as above. (We are also using the assumption $x\ge x_1$ to bound $1/\log x$.) By (4.14) and (4.15),

$$\begin{aligned}\int_0^\infty (\eta_+(t)-\eta_\circ(t))^2\log xt\,dt &\le \frac{547.56}{H^{7/2}}\log x + \frac{480.394}{H^{7/2}}\\ &\le 4.8398\cdot 10^{-6}\cdot\log x + 4.25\cdot 10^{-6}.\end{aligned}$$

We conclude that

(6.66)
$$\int_0^\infty \eta_+^2(t)\log xt\,dt = (0.640206 + O^*(1.2589\cdot 10^{-5}))\log x - 0.0210948 + O^*(8.7042\cdot 10^{-6})$$

We add to this the error term $2.2\cdot 10^{-11}\log x$ from (6.65), and simplify using the assumption $x\ge x_1$. We obtain:

$$\sum_{n=1}^\infty \Lambda(n)(\log n)\eta_+^2(n/x) = (0.6402 + O^*(2\cdot 10^{-5}))x\log x - 0.0211x. \quad (6.67)$$

7. The integral of the triple product over the minor arcs

7.1. The L_2 norm over arcs: variations on the large sieve for primes.

We are trying to estimate an integral $\int_{\mathbb{R}/\mathbb{Z}}|S(\alpha)|^3 d\alpha$. Rather than bound it by $|S|_\infty|S|_2^2$, we can use the fact that large ("major") values of $S(\alpha)$ have to be multiplied only by $\int_{\mathfrak{M}}|S(\alpha)|^2 d\alpha$, where $\mathfrak{M}$ is a union (small in measure) of minor arcs. Now, can we give an upper bound for $\int_{\mathfrak{M}}|S(\alpha)|^2 d\alpha$ better than $|S|_2^2 = \int_{\mathbb{R}/\mathbb{Z}}|S(\alpha)|^2 d\alpha$?

The first version of [Hel] gave an estimate on that integral using a technique due to Heath-Brown, which in turn rests on an inequality of Montgomery's ([Mon71, (3.9)]; see also, e.g., [IK04, Lem. 7.15]). The technique was communicated by

Heath-Brown to the present author, who communicated it to Tao ([Tao, Lem. 4.6] and adjoining comments). We will be able to do better than that estimate here.

The role played by Montgomery's inequality in Heath-Brown's method is played here by a result of Ramaré's ([Ram09, Thm. 2.1]; see also [Ram09, Thm. 5.2]). The following Proposition is based on Ramaré's result, or rather on one possible proof of it. Instead of using the result as stated in [Ram09], we will actually be using elements of the proof of [Bom74, Thm. 7A], credited to Selberg. Simply integrating Ramaré's inequality would give a non-trivial if slightly worse bound.

Proposition 7.1. *Let* $\{a_n\}_{n=1}^\infty$, $a_n \in \mathbb{C}$, *be supported on the primes. Assume that* $\{a_n\}$ *is in* $L_1 \cap L_2$ *and that* $a_n = 0$ *for* $n \leq \sqrt{x}$. *Let* $Q_0 \geq 1$, $\delta_0 \geq 1$ *be such that* $\delta_0 Q_0^2 \leq x/2$; *set* $Q = \sqrt{x/2\delta_0} \geq Q_0$. *Let*

$$\mathfrak{M} = \bigcup_{q \leq r} \bigcup_{\substack{a \bmod q \\ (a,q)=1}} \left(\frac{a}{q} - \frac{\delta_0 r}{qx}, \frac{a}{q} + \frac{\delta_0 r}{qx} \right). \tag{7.1}$$

Let $S(\alpha) = \sum_n a_n e(\alpha n)$ *for* $\alpha \in \mathbb{R}/\mathbb{Z}$. *Then*

$$\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha \leq \left(\max_{q \leq Q_0} \max_{s \leq Q_0/q} \frac{G_q(Q_0/sq)}{G_q(Q/sq)} \right) \sum_n |a_n|^2,$$

where

$$G_q(R) = \sum_{\substack{r \leq R \\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)}. \tag{7.2}$$

Proof. By (7.1),

$$\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha = \sum_{q \leq Q_0} \int_{-\frac{\delta_0 Q_0}{qx}}^{\frac{\delta_0 Q_0}{qx}} \sum_{\substack{a \bmod q \\ (a,q)=1}} \left| S\left(\frac{a}{q} + \alpha \right) \right|^2 d\alpha. \tag{7.3}$$

Thanks to the last equations of [Bom74, p. 24] and [Bom74, p. 25],

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} \left| S\left(\frac{a}{q} \right) \right|^2 = \frac{1}{\phi(q)} \sum_{\substack{q^* | q \\ (q^*, q/q^*)=1 \\ \mu^2(q/q^*)=1}} q^* \cdot \sideset{}{^*}\sum_{\chi \bmod q^*} \left| \sum_n a_n \chi(n) \right|^2$$

for every $q \leq \sqrt{x}$, where we use the assumption that n is prime and $> \sqrt{x}$ (and thus coprime to q) when $a_n \neq 0$. Hence

$$\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha = \sum_{q \leq Q_0} \sum_{\substack{q^*|q \\ (q^*,q/q^*)=1 \\ \mu^2(q/q^*)=1}} q^* \int_{-\frac{\delta_0 Q_0}{qx}}^{\frac{\delta_0 Q_0}{qx}} \frac{1}{\phi(q)} \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 d\alpha$$

$$= \sum_{q^* \leq Q_0} \frac{q^*}{\phi(q^*)} \sum_{\substack{r \leq Q_0/q^* \\ (r,q^*)=1}} \frac{\mu^2(r)}{\phi(r)} \int_{-\frac{\delta_0 Q_0}{q^* rx}}^{\frac{\delta_0 Q_0}{q^* rx}} \sideset{}{^*}\sum_{\chi \bmod q^*} \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 d\alpha$$

$$= \sum_{q^* \leq Q_0} \frac{q^*}{\phi(q^*)} \int_{-\frac{\delta_0 Q_0}{q^* x}}^{\frac{\delta_0 Q_0}{q^* x}} \sum_{\substack{r \leq \frac{Q_0}{q^*} \min\left(1, \frac{\delta_0}{|\alpha| x}\right) \\ (r,q^*)=1}} \frac{\mu^2(r)}{\phi(r)} \sideset{}{^*}\sum_{\chi \bmod q^*} \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 d\alpha$$

Here $|\alpha| \leq \delta_0 Q_0 / q^* x$ implies $(Q_0/q)\delta_0/|\alpha| x \geq 1$. Therefore,

$$\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha \leq \left(\max_{q^* \leq Q_0} \max_{s \leq Q_0/q^*} \frac{G_{q^*}(Q_0/sq^*)}{G_{q^*}(Q/sq^*)}\right) \cdot \Sigma, \tag{7.4}$$

where

$$\Sigma = \sum_{q^* \leq Q_0} \frac{q^*}{\phi(q^*)} \int_{-\frac{\delta_0 Q_0}{q^* x}}^{\frac{\delta_0 Q_0}{q^* x}} \sum_{\substack{r \leq \frac{Q}{q^*} \min\left(1, \frac{\delta_0}{|\alpha| x}\right) \\ (r,q^*)=1}} \frac{\mu^2(r)}{\phi(r)} \sideset{}{^*}\sum_{\chi \bmod q^*} \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 d\alpha$$

$$\leq \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\substack{r \leq Q/q \\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} \int_{-\frac{\delta_0 Q}{qrx}}^{\frac{\delta_0 Q}{qrx}} \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 d\alpha.$$

As stated in the proof of [Bom74, Thm. 7A],

$$\overline{\chi}(r)\chi(n)\tau(\overline{\chi})c_r(n) = \sum_{\substack{b=1 \\ (b,qr)=1}}^{qr} \overline{\chi}(b) e^{2\pi i n \frac{b}{qr}}$$

for χ primitive of modulus q. Here $c_r(n)$ stands for the Ramanujan sum

$$c_r(n) = \sum_{\substack{u \bmod r \\ (u,r)=1}} e^{2\pi n u/r}.$$

For n coprime to r, $c_r(n) = \mu(r)$. Since χ is primitive, $|\tau(\overline{\chi})| = \sqrt{q}$. Hence, for $r \leq \sqrt{x}$ coprime to q,

$$q \left|\sum_n a_n e(\alpha n)\chi(n)\right|^2 = \left|\sum_{\substack{b=1 \\ (b,qr)=1}}^{qr} \overline{\chi}(b) S\left(\frac{b}{qr}\right)\right|^2.$$

Thus,

$$\Sigma = \sum_{q\leq Q} \sum_{\substack{r\leq Q/q\\(r,q)=1}} \frac{\mu^2(r)}{\phi(rq)} \int_{-\frac{\delta_0 Q}{qrx}}^{\frac{\delta_0 Q}{qrx}} \sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{\substack{b=1\\(b,qr)=1}}^{qr} \overline{\chi}(b) S\left(\frac{b}{qr}\right)\right|^2 d\alpha$$

$$\leq \sum_{q\leq Q} \frac{1}{\phi(q)} \int_{-\frac{\delta_0 Q}{qx}}^{\frac{\delta_0 Q}{qx}} \sum_{\chi \bmod q} \left| \sum_{\substack{b=1\\(b,q)=1}}^{q} \overline{\chi}(b) S\left(\frac{b}{q}\right)\right|^2 d\alpha$$

$$= \sum_{q\leq Q} \int_{-\frac{\delta_0 Q}{qx}}^{\frac{\delta_0 Q}{qx}} \sum_{\substack{b=1\\(b,q)=1}}^{q} \left| S\left(\frac{b}{q}\right)\right|^2 d\alpha.$$

Let us now check that the intervals $(b/q - \delta_0 Q/qx, b/q + \delta_0 Q/qx)$ do not overlap. Since $Q = \sqrt{x/2\delta_0}$, we see that $\delta_0 Q/qx = 1/2qQ$. The difference between two distinct fractions b/q, b'/q' is at least $1/qq'$. For $q, q' \leq Q$, $1/qq' \geq 1/2qQ + 1/2Qq'$. Hence the intervals around b/q and b'/q' do not overlap. We conclude that

$$\Sigma \leq \int_{\mathbb{R}/\mathbb{Z}} \left| S\left(\frac{b}{q}\right)\right|^2 = \sum_n |a_n|^2,$$

and so, by (7.4), we are done. □

We will actually use Prop. 7.1 in the slightly modified form given by the following statement.

Proposition 7.2. *Let $\{a_n\}_{n=1}^\infty$, $a_n \in \mathbb{C}$, be supported on the primes. Assume that $\{a_n\}$ is in $L_1 \cap L_2$ and that $a_n = 0$ for $n \leq \sqrt{x}$. Let $Q_0 \geq 1$, $\delta_0 \geq 1$ be such that $\delta_0 Q_0^2 \leq x/2$; set $Q = \sqrt{x/2\delta_0} \geq Q_0$. Let $\mathfrak{M} = \mathfrak{M}_{\delta_0,Q_0}$ be as in (3.5).*

Let $S(\alpha) = \sum_n a_n e(\alpha n)$ for $\alpha \in \mathbb{R}/\mathbb{Z}$. Then

$$\int_{\mathfrak{M}_{\delta_0,Q_0}} |S(\alpha)|^2 d\alpha \leq \left(\max_{\substack{q\leq 2Q_0\\ q \text{ even}}} \max_{s\leq 2Q_0/q} \frac{G_q(2Q_0/sq)}{G_q(2Q/sq)}\right) \sum_n |a_n|^2,$$

where

$$G_q(R) = \sum_{\substack{r\leq R\\(r,q)=1}} \frac{\mu^2(r)}{\phi(r)}. \tag{7.5}$$

Proof. By (3.5),

$$\int_{\mathfrak{M}} |S(\alpha)|^2 d\alpha = \sum_{\substack{q\leq Q_0\\ q \text{ odd}}} \int_{-\frac{\delta_0 Q_0}{2qx}}^{\frac{\delta_0 Q_0}{2qx}} \sum_{\substack{a \bmod q\\(a,q)=1}} \left| S\left(\frac{a}{q}+\alpha\right)\right|^2 d\alpha$$

$$+ \sum_{\substack{q\leq Q_0\\ q \text{ even}}} \int_{-\frac{\delta_0 Q_0}{qx}}^{\frac{\delta_0 Q_0}{qx}} \sum_{\substack{a \bmod q\\(a,q)=1}} \left| S\left(\frac{a}{q}+\alpha\right)\right|^2 d\alpha.$$

We proceed as in the proof of Prop. 7.1. We still have (7.3). Hence $\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha$ equals

$$\sum_{\substack{q^* \leq Q_0 \\ q^* \text{ odd}}} \frac{q^*}{\phi(q^*)} \int_{-\frac{\delta_0 Q_0}{2q^* x}}^{\frac{\delta_0 Q_0}{2q^* x}} \sum_{\substack{r \leq \frac{Q_0}{q^*} \min\left(1, \frac{\delta_0}{2|\alpha| x}\right) \\ (r, 2q^*)=1}} \frac{\mu^2(r)}{\phi(r)} \sideset{}{^*}\sum_{\chi \bmod q^*} \left|\sum_n a_n e(\alpha n) \chi(n)\right|^2 d\alpha$$

$$+ \sum_{\substack{q^* \leq 2Q_0 \\ q^* \text{ even}}} \frac{q^*}{\phi(q^*)} \int_{-\frac{\delta_0 Q_0}{q^* x}}^{\frac{\delta_0 Q_0}{q^* x}} \sum_{\substack{r \leq \frac{2Q_0}{q^*} \min\left(1, \frac{\delta_0}{2|\alpha| x}\right) \\ (r, q^*)=1}} \frac{\mu^2(r)}{\phi(r)} \sideset{}{^*}\sum_{\chi \bmod q^*} \left|\sum_n a_n e(\alpha n) \chi(n)\right|^2 d\alpha.$$

(The sum with q odd and r even is equal to the first sum; hence the factor of 2 in front.) Therefore,

$$\begin{aligned}\int_{\mathfrak{M}} |S(\alpha)|^2 \, d\alpha \leq & \left(\max_{\substack{q^* \leq Q_0 \\ q^* \text{ odd}}} \max_{s \leq Q_0/q^*} \frac{G_{2q^*}(Q_0/sq^*)}{G_{2q^*}(Q/sq^*)}\right) \cdot 2\Sigma_1 \\ & + \left(\max_{\substack{q^* \leq 2Q_0 \\ q^* \text{ even}}} \max_{s \leq 2Q_0/q^*} \frac{G_{q^*}(2Q_0/sq^*)}{G_{q^*}(2Q/sq^*)}\right) \cdot \Sigma_2,\end{aligned} \tag{7.6}$$

where

$$\begin{aligned}\Sigma_1 &= \sum_{\substack{q \leq Q \\ q \text{ odd}}} \frac{q}{\phi(q)} \sum_{\substack{r \leq Q/q \\ (r,2q)=1}} \frac{\mu^2(r)}{\phi(r)} \int_{-\frac{\delta_0 Q}{2qrx}}^{\frac{\delta_0 Q}{2qrx}} \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum_n a_n e(\alpha n) \chi(n)\right|^2 d\alpha \\ &= \sum_{\substack{q \leq Q \\ q \text{ odd}}} \frac{q}{\phi(q)} \sum_{\substack{r \leq 2Q/q \\ (r,q)=1 \\ r \text{ even}}} \frac{\mu^2(r)}{\phi(r)} \int_{-\frac{\delta_0 Q}{qrx}}^{\frac{\delta_0 Q}{qrx}} \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum_n a_n e(\alpha n) \chi(n)\right|^2 d\alpha.\end{aligned}$$

$$\Sigma_2 = \sum_{\substack{q \leq 2Q \\ q \text{ even}}} \frac{q}{\phi(q)} \sum_{\substack{r \leq 2Q/q \\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} \int_{-\frac{\delta_0 Q}{qrx}}^{\frac{\delta_0 Q}{qrx}} \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum_n a_n e(\alpha n) \chi(n)\right|^2 d\alpha.$$

The two expressions within parentheses in (7.6) are actually equal.

Much as before, using [Bom74, Thm. 7A], we obtain that

$$\Sigma_1 \leq \sum_{\substack{q \leq Q \\ q \text{ odd}}} \frac{1}{\phi(q)} \int_{-\frac{\delta_0 Q}{2qx}}^{\frac{\delta_0 Q}{2qx}} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \left|S\left(\frac{b}{q}\right)\right|^2 d\alpha,$$

$$\Sigma_1 + \Sigma_2 \leq \sum_{\substack{q \leq 2Q \\ q \text{ even}}} \frac{1}{\phi(q)} \int_{-\frac{\delta_0 Q}{qx}}^{\frac{\delta_0 Q}{qx}} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \left|S\left(\frac{b}{q}\right)\right|^2 d\alpha.$$

Let us now check that the intervals of integration $(b/q - \delta_0 Q/2qx, b/q + \delta_0 Q/2qx)$ (for q odd), $(b/q - \delta_0 Q/qx, b/q + \delta_0 Q/qx)$ (for q even) do not overlap. Recall that $\delta_0 Q/qx = 1/2qQ$. The absolute value of the difference between two distinct fractions b/q, b'/q' is at least $1/qq'$. For $q, q' \leq Q$ odd, this is larger than $1/4qQ +$

$1/4Qq'$, and so the intervals do not overlap. For $q \le Q$ odd and $q' \le 2Q$ even (or vice versa), $1/qq' \ge 1/4qQ + 1/2Qq'$, and so, again the intervals do not overlap. If $q \le Q$ and $q' \le Q$ are both even, then $|b/q - b'/q'|$ is actually $\ge 2/qq'$. Clearly, $2/qq' \ge 1/2qQ + 1/2Qq'$, and so again there is no overlap. We conclude that

$$2\Sigma_1 + \Sigma_2 \le \int_{\mathbb{R}/\mathbb{Z}} \left|S\left(\frac{b}{q}\right)\right|^2 = \sum_n |a_n|^2.$$

□

7.2. **Bounding the quotient in the large sieve for primes.** The estimate given by Proposition 7.1 involves the quotient

$$\max_{q\le Q_0} \max_{s \le Q_0/q} \frac{G_q(Q_0/sq)}{G_q(Q/sq)}, \tag{7.7}$$

where G_q is as in (7.2). The appearance of such a quotient (at least for $s = 1$) is typical of Ramaré's version of the large sieve for primes; see, e.g., [Ram09]. We will see how to bound such a quotient in a way that is essentially optimal, not just asymptotically, but also in the ranges that are most relevant to us. (This includes, for example, $Q_0 \sim 10^6$, $Q \sim 10^{15}$.)

As the present work shows, Ramaré's work gives bounds that are, in some contexts, better than those of other large sieves for primes by a constant factor (approaching $e^\gamma = 1.78107\dots$). Thus, giving a fully explicit and nearly optimal bound for (7.7) is a task of clear general relevance, besides being needed for our main goal.

We will obtain bounds for $G_q(Q_0/sq)/G_q(Q/sq)$ when $Q_0 \le 2\cdot 10^{10}$, $Q \ge Q_0^2$. As we shall see, our bounds will be best when $s = q = 1$ – or, sometimes, when $s = 1$ and $q = 2$ instead.

Write $G(R)$ for $G_1(R) = \sum_{r\le R} \mu^2(r)/\phi(r)$. We will need several estimates for $G_q(R)$ and $G(R)$. As stated in [Ram95, Lemma 3.4],

$$G(R) \le \log R + 1.4709 \tag{7.8}$$

for $R \ge 1$. By [MV73, Lem. 7],

$$G(R) \ge \log R + 1.07 \tag{7.9}$$

for $R \ge 6$. There is also the trivial bound

$$\begin{aligned} G(R) &= \sum_{r\le R} \frac{\mu^2(r)}{\phi(r)} \le \sum_{r\le R} \frac{\mu^2(r)}{r} \prod_{p|r} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \sum_{r\le R} \frac{\mu^2(r)}{r} \prod_{p|r} \sum_{j\ge 1} \frac{1}{p^j} \ge \sum_{r\le R} \frac{1}{r} > \log R. \end{aligned} \tag{7.10}$$

The following bound, also well-known and easy,

$$G(R) \le \frac{q}{\phi(q)} G_q(R) \le G(Rq), \tag{7.11}$$

can be obtained by multiplying $G_q(R) = \sum_{r\le R:(r,q)=1} \mu^2(r)/\phi(r)$ term-by-term by $q/\phi(q) = \prod_{p|q}(1 + 1/\phi(p))$.

We will also use Ramaré's estimate from [Ram95, Lem. 3.4]:

$$G_d(R) = \frac{\phi(d)}{d} \left(\log R + c_E + \sum_{p|d} \frac{\log p}{p}\right) + O^*\left(7.284 R^{-1/3} f_1(d)\right) \tag{7.12}$$

for all $d \in \mathbb{Z}^+$ and all $R \geq 1$, where

$$f_1(d) = \prod_{p|d} (1 + p^{-2/3}) \left(1 + \frac{p^{1/3} + p^{2/3}}{p(p-1)}\right)^{-1} \tag{7.13}$$

and

$$c_E = \gamma + \sum_{p \geq 2} \frac{\log p}{p(p-1)} = 1.3325822\ldots \tag{7.14}$$

by [RS62, (2.11)].

If $R \geq 182$, then

$$\log R + 1.312 \leq G(R) \leq \log R + 1.354, \tag{7.15}$$

where the upper bound is valid for $R \geq 120$. This is true by (7.12) for $R \geq 4 \cdot 10^7$; we check (7.15) for $120 \leq R \leq 4 \cdot 10^7$ by a numerical computation.[10] Similarly, for $R \geq 200$,

$$\frac{\log R + 1.661}{2} \leq G_2(R) \leq \frac{\log R + 1.698}{2} \tag{7.16}$$

by (7.12) for $R \geq 1.6 \cdot 10^8$, and by a numerical computation for $200 \leq R \leq 1.6 \cdot 10^8$.

Write $\rho = (\log Q_0)/(\log Q) \leq 1$. We obtain immediately from (7.16) that

$$\frac{G_2(Q_0)}{G_2(Q)} \leq \frac{\log Q_0 + 1.661}{\log Q + 1.698} \tag{7.17}$$

for $Q, Q_0 \geq 200$.

Let us start by giving an easy bound, off from the truth by a factor of about e^γ (like some other versions of the large sieve). First, we need a simple explicit lemma.

Lemma 7.3. *Let $m \geq 1$, $q \geq 1$. Then*

$$\prod_{p|q \vee p \leq m} \frac{p}{p-1} \leq e^\gamma (\log(m + \log q) + 0.65771). \tag{7.18}$$

Proof. Let $\mathscr{P} = \prod_{p \leq m \vee p|q} p$. Then, by [RS75, (5.1)],

$$\mathscr{P} \leq q \prod_{p \leq m} p = q e^{\sum_{p \leq m} \log p} \leq q e^{(1+\epsilon_0)m},$$

where $\epsilon_0 = 0.001102$. Now, by [RS62, (3.42)],

$$\frac{n}{\phi(n)} \leq e^\gamma \log\log n + \frac{2.50637}{\log\log n} \leq e^\gamma \log\log x + \frac{2.50637}{\log\log x}$$

for all $x \geq n \geq 27$. Hence, if $qe^m \leq 27$,

$$\begin{aligned}\frac{\mathscr{P}}{\phi(\mathscr{P})} &\leq e^\gamma \left(\log((1+\epsilon_0)m + \log q) + \frac{2.50637}{\log(m + \log q)}\right) \\ &\leq e^\gamma \left(\log(m + \log q) + \epsilon_0 + \frac{2.50637/e^\gamma}{\log(m + \log q)}\right).\end{aligned}$$

Thus (7.18) holds when $m + \log q \geq 8.53$, since then $\epsilon_0 + (2.50637/e^\gamma)/\log(m + \log q) \leq 0.65771$. We verify all choices of $m, q \geq 1$ with $m + \log q \leq 8.53$ computationally; the worst case is that of $m = 1$, $q = 6$, which give the value 0.65771 in (7.18). □

[10] Using D. Platt's implementation [Pla11] of double-precision interval arithmetic based on Lambov's [Lam08] ideas.

Here is the promised easy bound.

Lemma 7.4. *Let $Q_0 \geq 1$, $Q \geq 182 Q_0$. Let $q \leq Q_0$, $s \leq Q_0/q$, q an integer. Then*

$$\frac{G_q(Q_0/sq)}{G_q(Q/sq)} \leq \frac{e^\gamma \log\left(\frac{Q_0}{sq} + \log q\right) + 1.172}{\log \frac{Q}{Q_0} + 1.31} \leq \frac{e^\gamma \log Q_0 + 1.172}{\log \frac{Q}{Q_0} + 1.31}.$$

Proof. Let $\mathscr{P} = \prod_{p \leq Q_0/sq \vee p|q} p$. Then

$$G_q(Q_0/sq) G_{\mathscr{P}}(Q/Q_0) \leq G_q(Q/sq)$$

and so

$$\frac{G_q(Q_0/sq)}{G_q(Q/sq)} \leq \frac{1}{G_{\mathscr{P}}(Q/Q_0)}. \tag{7.19}$$

Now the lower bound in (7.11) gives us that, for $d = \mathscr{P}$, $R = Q/Q_0$,

$$G_{\mathscr{P}}(Q/Q_0) \geq \frac{\phi(\mathscr{P})}{\mathscr{P}} G(Q/Q_0).$$

By Lem. 7.3,

$$\frac{\mathscr{P}}{\phi(\mathscr{P})} \leq e^\gamma \left(\log\left(\frac{Q_0}{sq} + \log q\right) + 0.658\right).$$

Hence, using (7.9), we get that

$$\frac{G_q(Q_0/sq)}{G_q(Q/sq)} \leq \frac{\mathscr{P}/\phi(\mathscr{P})}{G(Q/Q_0)} \leq \frac{e^\gamma \log\left(\frac{Q_0}{sq} + \log q\right) + 1.172}{\log \frac{Q}{Q_0} + 1.31}, \tag{7.20}$$

since $Q/Q_0 \geq 120$. Since

$$\left(\frac{Q_0}{sq} + \log q\right)' = -\frac{Q_0}{sq^2} + \frac{1}{q} = \frac{1}{q}\left(1 - \frac{Q_0}{sq}\right) \leq 0,$$

the rightmost expression of (7.20) is maximal for $q = 1$. □

We will use Lemma 7.4 when $Q_0 > 2 \cdot 10^{10}$, since then the numerical bounds we will derive are not available. As we will now see, we can also use Lemma 7.4 to obtain a bound that is useful when sq is large compared to Q_0, even when $Q_0 \leq 2 \cdot 10^{10}$.

Lemma 7.5. *Let $Q_0 \geq 1$, $Q \geq 200 Q_0$. Let $q \leq Q_0$, $s \leq Q_0/q$, q an even integer. Let $\rho = (\log Q_0)/\log Q \leq 2/3$. If*

$$\frac{2Q_0}{sq} \leq 1.1617 \cdot Q_0^{(1-\rho)e^{-\gamma}} - \log q,$$

then

$$\frac{G_q(2Q_0/sq)}{G_q(2Q/sq)} \leq \frac{\log Q_0 + 1.698}{\log Q}. \tag{7.21}$$

Proof. Apply Lemma 7.4. By (7.20), we see that (7.21) will hold provided that

$$\begin{aligned} e^\gamma \log\left(\frac{2Q_0}{sq} + \log q\right) + 1.172 &\leq \frac{\log \frac{Q}{Q_0} + 1.31}{\log Q} \cdot (\log Q_0 + 1.698) \\ &\leq \log Q_0 + 1.698 - \frac{(\log Q_0 + 1.698)(\log Q_0 - 1.31)}{\log Q}. \end{aligned} \tag{7.22}$$

Since $\rho = (\log Q_0)/\log Q$,

$$\begin{aligned}\log Q_0 + 1.698 - &\frac{(\log Q_0 + 1.698)(\log Q_0 - 1.31)}{\log Q}\\ &= \log Q_0 + 1.698 - \rho(\log Q_0 - 1.31) - \frac{1.698(\log Q_0 - 1.31)}{\log Q}\\ &\geq (1-\rho)\log Q_0 + 1.698 + 1.31\rho - 1.698\rho,\end{aligned}$$

and so (7.22) will hold provided that

$$e^\gamma \log\left(\frac{2Q_0}{sq} + \log q\right) + 1.172 \leq (1-\rho)\log Q_0 + 1.698 + 1.31\rho - 1.698\rho.$$

For all $\rho \in [0, 2/3]$,

$$1.698 + 1.31\rho - 1.698\rho - 1.172 \geq 0.526 - \frac{2}{3}0.388 \geq 0.267.$$

Hence it is enough that

$$\frac{2Q_0}{sq} + \log q \leq e^{e^{-\gamma}((1-\rho)\log Q_0 + 0.267)} = c \cdot Q_0^{(1-\rho)e^{-\gamma}},$$

where $c = \exp(\exp(-\gamma)\cdot 0.153) = 1.16172\ldots$. □

Proposition 7.6. *Let $Q_0 \geq 10^5$, $Q \geq 200Q_0$. Let $\rho = (\log Q_0)/\log Q$. Assume $\rho \leq \rho_1 = 0.55$. Then, for every even and positive $q \leq 2Q_0$ and every $s \in [1, 2Q_0/q]$,*

$$\frac{G_q(2Q_0/sq)}{G_q(2Q/sq)} \leq \frac{\log Q_0 + 1.698}{\log Q}. \tag{7.23}$$

Proof. Define $\mathrm{err}_{q,R}$ so that

$$G_q(R) = \frac{\phi(q)}{q}\left(\log R + c_E + \sum_{p|q}\frac{\log p}{p}\right) + \mathrm{err}_{q,R}. \tag{7.24}$$

By (7.11) and (7.16), $G_q(2Q/sq) \geq (\phi(q)/q)(\log 2Q/sq + 1.661)$. Hence, (7.23) will hold when

$$\log\frac{2Q_0}{sq} + c_E + \sum_{p|q}\frac{\log p}{p} + \frac{q}{\phi(q)}\mathrm{err}_{q,\frac{2Q_0}{sq}} \leq (\log 2Q/sq + 1.661)\frac{\log Q_0 + 1.698}{\log Q}. \tag{7.25}$$

We know that $\log sq \geq \log 2$. The right side of (7.25) equals

$$\log Q_0 + 1.698 - (\log\frac{sq}{2} - 1.661)\frac{\log Q_0 + 1.698}{\log Q}.$$

Note that

$$\frac{\log Q_0 + 1.698}{\log Q} = \rho + \frac{1.698}{\log Q} = \rho\left(1 + \frac{1.698}{\log Q_0}\right).$$

Thus, (7.25) will be true provided that

$$\begin{aligned}\log sq - \rho\left(1 + \frac{1.698}{\log Q_0}\right)(\log\frac{sq}{2} - 1.661) + (1.698 - c_E)\\ \geq \sum_{p|q}\frac{\log p}{p} + \frac{q}{\phi(q)}\mathrm{err}_{q,\frac{2Q_0}{sq}}\end{aligned}$$

Moreover, if this is true for $\rho = \rho_1$, it is true for all $\rho \leq \rho_1$. Thus, it is enough to check that

$$\log sq - \rho_1 \left(1 + \frac{1.698}{\log Q_{0,\min}}\right)(\log \frac{sq}{2} - 1.661) + 0.365 \geq \sum_{p|q} \frac{\log p}{p} + \frac{q}{\phi(q)} \mathrm{err}_{q,\frac{2Q_0}{sq}}, \tag{7.26}$$

where $Q_{0,\min} = 10^5$. Since $Q_{0,\min} = 10^5$ and $\rho_1 \leq 0.87$, we know that $\rho_1(1 + 1.698/\log Q_{0,\min}) < 1.1475\rho_1 < 1$. Now all that remains to do is to take the maximum $m_{q,R,1}$ of $\mathrm{err}_{q,R}$ over all R satisfying

$$R > 1.1617 \cdot \max(Rq, Q_{0,\min})^{(1-\rho_1)e^{-\gamma}} - \log q \tag{7.27}$$

(since all smaller R are covered by Lemma 7.5) and verify that

$$m_{q,R,1} \leq \frac{\phi(q)}{q} \kappa(q) \tag{7.28}$$

for

$$\kappa(q) = (1 - 1.1475\rho_1) \log q + 2.70135\rho_1 + 0.365 - \sum_{p|q} \frac{\log p}{p}.$$

(Values of s larger than 1 make the conditions stricter and the conclusions weaker, thus making the task only easier.)

Ramaré's bound (7.12) implies that

$$\mathrm{err}_{q,R} \leq 7.284 R^{-1/3} f_1(q),$$

with $f_1(q)$ as in (7.13). This is enough when

$$R \geq \lambda(q) = \left(\frac{q}{\phi(q)} \frac{7.284 f_1(q)}{\kappa(q)}\right)^3. \tag{7.29}$$

The first question is: for which q does (7.27) imply (7.29)? It is easy to see that (7.27) implies

$$R^{1-(1-\rho_1)e^{-\gamma}} > 1.1617 q^{(1-\rho_1)e^{-\gamma}} - \frac{\log q}{R^{(1-\rho_1)e^{-\gamma}}} > 1.1617 q^{(1-\rho_1)e^{-\gamma}} - \log q$$

and so, assuming $1.1617 q^{(1-\rho_1)e^{-\gamma}} - \log q > 0$, we get that $R > \varpi(q)$, where

$$\varpi(q) = \left(1.1617 q^{(1-\rho_1)e^{-\gamma}} - \frac{\log q}{(1.1617 q^{(1-\rho_1)e^{-\gamma}} - \log q)^{\frac{1}{1-(1-\rho_1)e^{-\gamma}}}}\right)^{\frac{1}{1-(1-\rho_1)e^{-\gamma}}}. \tag{7.30}$$

Thus, if $\varpi(q)$ is greater than the quantity $\lambda(q)$ given in (7.29), we are done.

Now, $(p/(p-1)) \cdot f_1(p)$ and $p \to (\log p)/p$ are decreasing functions of p for $p \geq 3$. Hence, for even $q < \prod_{p \leq p_0} p$, p_0 a prime,

$$\kappa(q) \geq (1 - 1.1475\rho_1) \log q + 2.271 - \sum_{p < p_0} \frac{\log p}{p} \tag{7.31}$$

and

$$\lambda(q) \leq \left(\prod_{p<p_0} \frac{p}{p-1} \cdot \frac{7.284 \cdot \prod_{p<p_0} f_1(p)}{(1 - 1.1475\rho_1) \log q + 2.70135\rho_1 + 0.365 - \sum_{p<p_0} \frac{\log p}{p}}\right)^3. \tag{7.32}$$

For instance, since $\rho_1 = 0.55$,

$$\lambda(q) \leq \begin{cases} \left(\frac{51.273}{0.3688 \log q + 0.3414}\right)^3 & \text{for } q < \prod_{p\leq 23} p = 223092870, \\ \left(\frac{58.958}{0.3688 \log q + 0.2051}\right)^3 & \text{for } q < \prod_{p\leq 29} p = 6469693230, \\ \left(\frac{66.508}{0.3688 \log q + 0.0889}\right)^3 & \text{for } q < \prod_{p\leq 31} p = 200560490130. \end{cases}$$

Since $q \mapsto 1.1617 q^{(1-\rho_1)e^{-\gamma}} - \log q$ is increasing for $q > 128$, so is $q \mapsto \varpi(q)$. A comparison for $q = 3.59 \cdot 10^7$, $q = \prod_{p\leq 23} p$ and $q = \prod_{p\leq 29} p$ shows that $\varpi(q) > \lambda(q)$ for $q \in [3.59 \cdot 10^7, \prod_{p\leq 31} p)$. For $q \geq \prod_{p\leq 31} p$, we know that

$$\left(1.1617 q^{(1-0.55)e^{-\gamma}} - \log q\right)^{\frac{1}{1-(1-0.55)\cdot e^{-\gamma}}} \geq 298.03 \log q$$

and so

$$\begin{aligned} \varpi(q) &\geq \left(1.1617 q^{(1-0.55)e^{-\gamma}} - \frac{1}{298.03}\right)^{\frac{1}{1-(1-0.55)\cdot e^{-\gamma}}} \\ &\geq \left(1.1616 q^{(1-0.55)e^{-\gamma}}\right)^{\frac{1}{1-(1-0.55)\cdot e^{-\gamma}}} \geq 1.222 q^{0.338}. \end{aligned}$$

By [RS62, (3.14), (3.24), (3.30)] and a numerical computation for $p_1 < 10^6$,

$$\sum_{p\leq p_1} \frac{\log p}{p} < \log p_1, \quad \log\left(\prod_{p\leq p_1} p\right) = \sum_{p\leq p_1} \log p > 0.8 p_1,$$
$$\prod_{p\leq p_1} \frac{p}{p-1} < e^{\gamma}\left(1 + \frac{1}{\log^2 p_1}\right)(\log p_1) < 1.94 \log p_1$$

for all $p_1 \geq 31$. This implies, in particular, that, for $q = \prod_{p\leq p_1} p$, $\varpi(q) > 1.222 e^{0.8\cdot 0.338 p_1} = 1.222 e^{0.2704 p_1}$.

We also have

$$\prod_{p\leq 31}\left(1 + \frac{p^{1/3} + p^{2/3}}{p(p-1)}\right)^{-1} \leq 0.1489$$

and

$$\begin{aligned} \log\left(f_1\left(\prod_{p\leq p_1} p\right)\right) &= \sum_{p\leq p_1} \log(1 + p^{-2/3}) + \log \prod_{p\leq 31}\left(1 + \frac{p^{1/3} + p^{2/3}}{p(p-1)}\right)^{-1} \\ &\leq 0.729 p_1^{1/3} + \log 0.1489. \end{aligned}$$

Hence

$$\begin{aligned} \lambda(q) &\leq \left(1.94 \log p_1 \cdot \frac{7.284 \cdot 0.1489 e^{0.729 p_1^{1/3}}}{0.3688 \cdot 0.8 p_1 - \log p_1}\right)^3 \leq \left(1.265 e^{0.729 p_1^{1/3}}\right)^3 \\ &\leq 2.025 e^{0.2217 p_1} \end{aligned}$$

for $p_1 \geq 31$. Since $2.025 e^{0.2217 p_1} < 1.222 e^{0.2704 p_1}$ for $p_1 \geq 31$, it follows that $\varpi(q) > \lambda(q)$ for $q = \prod_{p\leq p_1} p$, $p_1 \geq 31$. Looking at (7.31) and (7.32), we see that this implies that $\varpi(q) > \lambda(q)$ holds for all $q \geq \prod_{p\leq 31} p$. Together with our previous calculations, this gives us that $\varpi(q) > \lambda(q)$ for all $q \geq 3.59 \cdot 10^7$.

Now, for $q < 3.59 \cdot 10^7$ even, we need to check that the maximum $m_{q,R,1}$ of $\text{err}_{q,R}$ over all $\varpi(q) \leq R < \lambda(q)$ satisfies (7.28). Since $\log R$ is increasing on R and

$G_q(R)$ depends only on $\lfloor R \rfloor$, we can tell from (7.24) that, since we are taking the maximum of $\mathrm{err}_{q,R}$, it is enough to check integer values of R. We check all integers R in $[\varpi(q), \lambda(q))$ for all even $q < 3.59 \cdot 10^7$ by an explicit computation.[11] □

Finally, we have the trivial bound

$$\frac{G_q(Q_0/sq)}{G_q(Q/sq)} \leq 1, \tag{7.33}$$

which we shall use for Q_0 close to Q.

Corollary 7.7. *Let $\{a_n\}_{n=1}^\infty$, $a_n \in \mathbb{C}$, be supported on the primes. Assume that $\{a_n\}$ is in $L_1 \cap L_2$ and that $a_n = 0$ for $n \leq \sqrt{x}$. Let $Q_0 \geq 10^5$, $\delta_0 \geq 1$ be such that $(200Q_0)^2 \leq x/2\delta_0$; set $Q = \sqrt{x/2\delta_0}$ and $\rho = (\log Q_0)/\log Q$. Let $\mathfrak{M}_{\delta_0,Q_0}$ be as in (3.5).*

Let $S(\alpha) = \sum_n a_n e(\alpha n)$ for $\alpha \in \mathbb{R}/\mathbb{Z}$. Then, if $\rho \leq 0.55$,

$$\int_{\mathfrak{M}_{\delta_0,Q_0}} |S(\alpha)|^2 \, d\alpha \leq \frac{\log Q_0 + 1.698}{\log Q} \int_{\mathbb{R}/\mathbb{Z}} |S(\alpha)|^2 \, d\alpha.$$

Here, of course, $\int_{\mathbb{R}/\mathbb{Z}} |S(\alpha)|^2 \, d\alpha = \sum_n |a_n|^2$ (Plancherel). If $\rho > 0.55$, we will use the trivial bound

$$\int_{\mathfrak{M}_{\delta_0,r}} |S(\alpha)|^2 \, d\alpha \leq \int_{\mathbb{R}/\mathbb{Z}} |S(\alpha)|^2 \, d\alpha \tag{7.34}$$

Proof. Immediate from Prop. 7.2, Lem. 7.4 and Prop. 7.6. □

7.3. Putting together ℓ_2 bounds over arcs and ℓ_∞ bounds.

First, we need a simple lemma – essentially a way to obtain upper bounds by means of summation by parts.

Lemma 7.8. *Let $f, g : \{a, a+1, \ldots, b\} \to \mathbb{R}_0^+$, where $a, b \in \mathbb{Z}^+$. Assume that, for all $x \in [a, b]$,*

$$\sum_{a \leq n \leq x} f(n) \leq F(x), \tag{7.35}$$

where $F : [a, b] \to \mathbb{R}$ is continuous, piecewise differentiable and non-decreasing. Then

$$\sum_{n=a}^{b} f(n) \cdot g(n) \leq (\max_{n \geq a} g(n)) \cdot F(a) + \int_a^b (\max_{n \geq u} g(n)) \cdot F'(u) du.$$

Proof. Let $S(n) = \sum_{m=a}^{n} f(m)$. Then, by partial summation,

$$\sum_{n=a}^{b} f(n) \cdot g(n) \leq S(b) g(b) + \sum_{n=a}^{b-1} S(n)(g(n) - g(n+1)). \tag{7.36}$$

[11] Here, as elsewhere in this section, numerical computations were carried out by the author in C; all floating-point operations used D. Platt's interval arithmetic package.

Let $h(x) = \max_{x\leq n\leq b} g(n)$. Then h is non-increasing. Hence (7.35) and (7.36) imply that

$$\begin{aligned}\sum_{n=a}^{b} f(n)g(n) &\leq \sum_{n=a}^{b} f(n)h(n)\\ &\leq S(b)h(b) + \sum_{n=a}^{b-1} S(n)(h(n)-h(n+1))\\ &\leq F(b)h(b) + \sum_{n=a}^{b-1} F(n)(h(n)-h(n+1)).\end{aligned}$$

In general, for $\alpha_n \in \mathbb{C}$, $A(x) = \sum_{a\leq n\leq x} \alpha_n$ and F continuous and piecewise differentiable on $[a,x]$,

$$\sum_{a\leq n\leq x} \alpha_n F(x) = A(x)F(x) - \int_a^x A(u)F'(u)du. \qquad (\textit{Abel summation})$$

Applying this with $\alpha_n = h(n)-h(n+1)$ and $A(x) = \sum_{a\leq n\leq x} \alpha_n = h(a)-h(\lfloor x\rfloor+1)$, we obtain

$$\begin{aligned}&\sum_{n=a}^{b-1} F(n)(h(n)-h(n+1))\\ &\quad = (h(a)-h(b))F(b-1) - \int_a^{b-1} (h(a)-h(\lfloor u\rfloor+1))F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b-1) + \int_a^{b-1} h(\lfloor u\rfloor+1)F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b-1) + \int_a^{b-1} h(u)F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b) + \int_a^{b} h(u)F'(u)du,\end{aligned}$$

since $h(\lfloor u\rfloor+1) = h(u)$ for $u \notin \mathbb{Z}$. Hence

$$\sum_{n=a}^{b} f(n)g(n) \leq h(a)F(a) + \int_a^b h(u)F'(u)du.$$

□

We will now see our main application of Lemma 7.8. We have to bound an integral of the form $\int_{\mathfrak{M}_{\delta_0,r}} |S_1(\alpha)|^2|S_2(\alpha)|d\alpha$, where $\mathfrak{M}_{\delta_0,r}$ is a union of arcs defined as in (3.5). Our inputs are (a) a bound on integrals of the form $\int_{\mathfrak{M}_{\delta_0,r}} |S_1(\alpha)|^2 d\alpha$, (b) a bound on $|S_2(\alpha)|$ for $\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r}$. The input of type (a) is what we derived in §7.1 and §7.2; the input of type (b) is a minor-arcs bound, and as such is the main subject of [Hel].

Proposition 7.9. *Let $S_1(\alpha) = \sum_n a_n e(\alpha n)$, $a_n \in \mathbb{C}$, $\{a_n\}$ in L^1. Let $S_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be continuous. Define $\mathfrak{M}_{\delta_0,r}$ as in (3.5).*

Let r_0 be a positive integer not greater than r_1. Let $H : [r_0, r_1] \to \mathbb{R}^+$ be a non-decreasing continuous function, continuous and differentiable almost everywhere, *such that*

$$\frac{1}{\sum |a_n|^2} \int_{\mathfrak{M}_{\delta_0,r+1}} |S_1(\alpha)|^2 d\alpha \leq H(r) \tag{7.37}$$

for some $\delta_0 \leq x/2r_1^2$ and all $r \in [r_0, r_1]$. Assume, moreover, that $H(r_1) = 1$. Let $g : [r_0, r_1] \to \mathbb{R}^+$ be a non-increasing function such that

$$\max_{\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r}} |S_2(\alpha)| \leq g(r) \tag{7.38}$$

for all $r \in [r_0, r_1]$ and δ_0 as above.

Then

$$\begin{aligned} \frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r_0}} |S_1(\alpha)|^2 |S_2(\alpha)| d\alpha \\ \leq g(r_0) \cdot (H(r_0) - I_0) + \int_{r_0}^{r_1} g(r) H'(r) dr, \end{aligned} \tag{7.39}$$

where

$$I_0 = \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{M}_{\delta_0,r_0}} |S_1(\alpha)|^2 d\alpha. \tag{7.40}$$

The condition $\delta_0 \leq x/2r_1^2$ is there just to ensure that the arcs in the definition of $\mathfrak{M}_{\delta_0,r}$ do not overlap for $r \leq r_1$.

Proof. For $r_0 \leq r < r_1$, let

$$f(r) = \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{M}_{\delta_0,r+1} \setminus \mathfrak{M}_{\delta_0,r}} |S_1(\alpha)|^2 d\alpha.$$

Let

$$f(r_1) = \frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r_1}} |S_1(\alpha)|^2 d\alpha.$$

Then, by (7.38),

$$\frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r_0}} |S_1(\alpha)|^2 |S_2(\alpha)| d\alpha \leq \sum_{r=r_0}^{r_1} f(r) g(r).$$

By (7.37),

$$\begin{aligned} \sum_{r_0 \leq r \leq x} f(r) &= \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{M}_{\delta_0,x+1} \setminus \mathfrak{M}_{\delta_0,r_0}} |S_1(\alpha)|^2 d\alpha \\ &= \left(\frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{M}_{\delta_0,x+1}} |S_1(\alpha)|^2 d\alpha \right) - I_0 \leq H(x) - I_0 \end{aligned} \tag{7.41}$$

for $x \in [r_0, r_1)$. Moreover,

$$\begin{aligned} \sum_{r_0 \leq r \leq r_1} f(r) &= \frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\delta_0,r_0}} |S_1(\alpha)|^2 \\ &= \left(\frac{1}{\sum_n |a_n|^2} \int_{\mathbb{R}/\mathbb{Z}} |S_1(\alpha)|^2 \right) - I_0 = 1 - I_0 = H(r_1) - I_0. \end{aligned}$$

We let $F(x) = H(x) - I_0$ and apply Lemma 7.8 with $a = r_0$, $b = r_1$. We obtain that

$$\sum_{r=r_0}^{r_1} f(r)g(r) \leq (\max_{r\geq r_0} g(r))F(r_0) + \int_{r_0}^{r_1} (\max_{r\geq u} g(r))F'(u)\, du$$
$$\leq g(r_0)(H(r_0) - I_0) + \int_{r_0}^{r_1} g(u)H'(u)\, du.$$

□

Theorem 7.10 (Total of minor arcs)**.** *Let $x \geq 10^{24} \cdot \varkappa$, where $4 \leq \varkappa \leq 1750$. Let*

$$S_\eta(\alpha, x) = \sum_n \Lambda(n)e(\alpha n)\eta(n/x). \tag{7.42}$$

Let $\eta_*(t) = (\eta_2 *_M \varphi)(\varkappa t)$, where η_2 is as in (4.36) and $\varphi : [0,\infty) \to [0,\infty)$ is continuous and in L^1. Let $\eta_+ : [0,\infty) \to [0,\infty)$ be a bounded, piecewise differentiable function with $\lim_{t\to\infty} \eta_+(t) = 0$. Let $\mathfrak{M}_{\delta_0,r}$ be as in (3.5) with $\delta_0 = 8$. Let $10^5 \leq r_0 < r_1$, where $r_1 = (2/3)(x/\varkappa)^{0.55/2}$.

Let

$$Z_{r_0} = \int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha, x)||S_{\eta_+}(\alpha, x)|^2 d\alpha.$$

Then

$$Z_{r_0} \leq \left(\sqrt{\frac{|\varphi|_1 x}{\varkappa}(M+T)} + \sqrt{S_{\eta_*}(0,x)\cdot E}\right)^2,$$

where

$$\begin{aligned} S &= \sum_{p>\sqrt{x}} (\log p)^2\eta_+^2(n/x),\\ T &= C_{\varphi,3}(\log x)\cdot(S - (\sqrt{J} - \sqrt{E})^2),\\ J &= \int_{\mathfrak{M}_{8,r_0}} |S_{\eta_+}(\alpha, x)|^2\, d\alpha,\\ E &= \left((C_{\eta_+,0} + C_{\eta_+,2})\log x + (2C_{\eta_+,0} + C_{\eta_+,1})\right)\cdot x^{1/2}, \end{aligned} \tag{7.43}$$

$$\begin{aligned} C_{\eta_+,0} &= 0.7131\int_0^\infty \frac{1}{\sqrt{t}}(\sup_{r\geq t}\eta_+(r))^2 dt,\\ C_{\eta_+,1} &= 0.7131\int_0^\infty \frac{\log t}{\sqrt{t}}(\sup_{r\geq t}\eta_+(r))^2 dt,\\ C_{\eta_+,2} &= 0.51941|\eta_+|_\infty^2,\\ C_{\varphi,3}(K) &= \frac{1.04488}{|\varphi|_1}\int_0^{1/K} |\varphi(w)|dw \end{aligned} \tag{7.44}$$

and

$$\begin{aligned} M &= g(r_0)\cdot\left(\frac{(\log(r_0+1)) + 1.698}{\log\sqrt{x/16}}\cdot S - (\sqrt{J}-\sqrt{E})^2\right)\\ &\quad + \left(\frac{2}{\log\frac{x}{16}}\int_{r_0}^{r_1}\frac{g(r)}{r}dr + 0.45g(r_1)\right)\cdot S \end{aligned} \tag{7.45}$$

where $g(r) = g_{x/\varkappa,\varphi}(r)$ with $K = \log(x/\varkappa)$ (see (4.46)).

Proof. Let $y = x/\varkappa$. Let $Q = (3/4)y^{2/3}$, as in [Hel, Main Thm.] (applied with y instead of x). Let $\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{8,r}$, where $r \geq r_0$ and y is used instead of x to define $\mathfrak{M}_{8,r}$ (see (3.5)). There exists an approximation $2\alpha = a/q + \delta/y$ with $q \leq Q$, $|\delta|/y \leq 1/qQ$. Thus, $\alpha = a'/q' + \delta/2y$, where either $a'/q' = a/2q$ or $a'/q' = (a+q)/2q$ holds. (In particular, if q' is odd, then $q' = q$; if q' is even, then q' may be q or $2q$.)

There are three cases:

(1) $q \leq r$. Then either (a) q' is odd and $q' \leq r$ or (b) q' is even and $q' \leq 2r$. Since α is not in $\mathfrak{M}_{8,r}$, then, by definition (3.5), $|\delta|/y \geq \delta_0 r/q = 8r/q$. In particular, $|\delta| \geq 8$.

Thus, by Prop. 4.5 and Lemma 4.6,

$$|S_{\eta_*}(\alpha, x)| = |S_{\eta_2 *_M \phi}(\alpha, y)| \leq g_{y,\varphi}\left(\frac{|\delta|}{8} q\right) \cdot |\varphi|_1 y \leq g_{y,\varphi}(r) \cdot |\varphi|_1 y. \tag{7.46}$$

(2) $r < q \leq y^{1/3}/6$. Then, by Prop. 4.5 and Lemma 4.6,

$$|S_{\eta_*}(\alpha, x)| = |S_{\eta_2 *_M \phi}(\alpha, y)| \leq g_{y,\varphi}\left(\max\left(\frac{|\delta|}{8}, 1\right) q\right) \cdot |\varphi|_1 y \leq g_{y,\varphi}(r) \cdot |\varphi|_1 y. \tag{7.47}$$

(3) $q > y^{1/3}/6$. Again by Prop. 4.5,

$$|S_{\eta_*}(\alpha, x)| = |S_{\eta_2 *_M \phi}(\alpha, y)| \leq \left(h\left(\frac{y}{K}\right) |\varphi|_1 + C_{\varphi,3}(K)\right) y, \tag{7.48}$$

where $h(x)$ is as in (4.42). (Note that $C_{\varphi,3}(K)$, as in (7.44), equals $C_{\varphi,0,K}/|\phi|_1$, where $C_{\varphi,0,K}$ is as in (4.48).) We set $K = \log y$. Since $y = x/\kappa \geq 10^{24}$, it follows that $y/K = y/\log y > 2.16 \cdot 10^{20}$.

Let

$$r_1 = \frac{2}{3} y^{\frac{0.55}{2}}, \qquad g(r) = \begin{cases} g_{x,\varphi}(r) & \text{if } r \leq r_1, \\ g_{x,\varphi}(r_1) & \text{if } r > r_1. \end{cases}$$

Since $\varkappa > 4$, we see that $r_1 \leq (x/16)^{0.55/2}$. By Lemma 4.6, $g(r)$ is a decreasing function; moreover, by Lemma 4.7, $g_{y,\phi}(r_1) \geq h(y/\log y)$, and so $g(r) \geq h(y/\log y)$ for all r. Thus, we have shown that

$$|S_{\eta_*}(y, \alpha)| \leq (g(r) + C_{\varphi,3}(\log y)) \cdot |\varphi|_1 y \tag{7.49}$$

for all $\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{8,r}$.

We first need to undertake the fairly dull task of getting non-prime or small n out of the sum defining $S_{\eta_+}(\alpha, x)$. Write

$$S_{1,\eta_+}(\alpha, x) = \sum_{p > \sqrt{x}} (\log p) e(\alpha p) \eta_*(p/x),$$

$$S_{2,\eta_+}(\alpha, x) = \sum_{\substack{n \text{ non-prime} \\ n > \sqrt{x}}} \Lambda(n) e(\alpha n) \eta_+(n/x) + \sum_{n \leq \sqrt{x}} \Lambda(n) e(\alpha n) \eta_+(n/x).$$

By the triangle inequality (with weights $|S_{\eta_+}(\alpha, x)|$),

$$\sqrt{\int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha, x)| |S_{\eta_+}(\alpha, x)|^2 d\alpha}$$
$$\leq \sum_{j=1}^{2} \sqrt{\int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha, x)| |S_{j,\eta_+}(\alpha, x)|^2 d\alpha}.$$

Clearly,

$$\begin{aligned}&\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha,x)||S_{2,\eta_+}(\alpha,x)|^2 d\alpha\\ &\leq \max_{\alpha\in(\mathbb{R}/\mathbb{Z})} |S_{\eta_*}(\alpha,x)|\cdot\int_{\mathbb{R}/\mathbb{Z}} |S_{2,\eta_+}(\alpha,x)|^2 d\alpha\\ &\leq \sum_{n=1}^\infty \Lambda(n)\eta_*(n/x)\cdot\left(\sum_{n \text{ non-prime}} \Lambda(n)^2\eta_+(n/x)^2 + \sum_{n\leq\sqrt{x}} \Lambda(n)^2\eta_+(n/x)^2\right).\end{aligned}$$

Let $\overline{\eta_+}(z) = \sup_{t\geq z}\eta_+(t)$. Since $\eta_+(t)$ tends to 0 as $t\to\infty$, so does $\overline{\eta_+}$. By [RS62, Thm. 13], partial summation and integration by parts,

$$\begin{aligned}&\sum_{n \text{ non-prime}} \Lambda(n)^2\eta_+(n/x)^2 \leq \sum_{n \text{ non-prime}} \Lambda(n)^2\overline{\eta_+}(n/x)^2\\ &\leq \int_1^\infty \left(\sum_{\substack{n\leq t\\ n \text{ non-prime}}} \Lambda(n)^2\right)\left(\overline{\eta_+}^2(t/x)\right)' dt \leq \int_1^\infty (\log t)\cdot 1.4262\sqrt{t}\left(\overline{\eta_+}^2(t/x)\right)' dt\\ &\leq 0.7131\int_1^\infty \frac{\log e^2 t}{\sqrt{t}}\cdot\overline{\eta_+}^2\left(\frac{t}{x}\right)dt \leq \left(0.7131\int_0^\infty \frac{2+\log tx}{\sqrt{t}}\overline{\eta_+}^2(t)dt\right)\sqrt{x},\end{aligned}$$

while, by [RS62, Thm. 12],

$$\begin{aligned}\sum_{n\leq\sqrt{x}} \Lambda(n)^2\eta_+(n/x)^2 &\leq \frac{1}{2}|\eta_+|_\infty^2(\log x)\sum_{n\leq r_1}\Lambda(n)\\ &\leq 0.51941|\eta_+|_\infty^2\cdot\sqrt{x}\log x.\end{aligned}$$

This shows that

$$\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha,x)||S_{2,\eta_+}(\alpha,x)|^2 d\alpha \leq \sum_{n=1}^\infty \Lambda(n)\eta_*(n/x)\cdot E = S_{\eta_*}(0,x)\cdot E,$$

where E is as in (7.43).

It remains to bound

$$\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha,x)||S_{1,\eta_+}(\alpha,x)|^2 d\alpha. \tag{7.50}$$

We wish to apply Prop. 7.9. Corollary 7.7 gives us an input of type (7.37); we have just derived a bound (7.43) that provides an input of type (7.38). More precisely, by Corollary 7.7, (7.37) holds with

$$H(r) = \begin{cases}\frac{(\log(r+1))+1.698}{\log\sqrt{x/16}} & \text{if } r < r_1,\\ 1 & \text{if } r\geq r_1.\end{cases}$$

Since $r_1 = (2/3)y^{0.275}$ and $\varkappa\leq 1750$,

$$\begin{aligned}\lim_{r\to r_1^+} H(r) - \lim_{r\to r_1^-} H(r) &= 1 - \frac{\log((2/3)(x/\varkappa)^{0.275}+1)+1.698}{\log\sqrt{x/16}}\\ &\leq 1 - \left(\frac{0.275}{0.5} + \frac{\log\frac{2}{3}+1.698+0.275\log\frac{16}{\varkappa}}{\log\sqrt{x/16}}\right) \leq 1 - \frac{0.275}{0.5} = 0.45.\end{aligned}$$

We also have (7.38) with

$$(g(r) + C_{\varphi,3}(\log y)) \cdot |\varphi|_1 y \tag{7.51}$$

instead of $g(r)$ (by (7.49)). Here (7.51) is a decreasing function of r because $g(r)$ is, as we already checked. Hence, Prop. 7.9 gives us that (7.50) is at most

$$g(r_0)\cdot(H(r_0) - I_0) + (1 - I_0)\cdot C_{\varphi,3}(\log x) + \frac{1}{\log\sqrt{x/16}}\int_{r_0}^{r_1}\frac{g(r)}{r+1}dr + \frac{9g(r_1)}{20} \tag{7.52}$$

times $|\varphi|_1 y \cdot \sum_{p>\sqrt{x}}(\log p)^2\eta_*^2(p/x)$, where

$$I_0 = \frac{1}{\sum_{p>\sqrt{x}}(\log p)^2\eta_+^2(n/x)}\int_{\mathfrak{M}_{8,r_0}} |S_{1,\eta_+}(\alpha,x)|^2\, d\alpha. \tag{7.53}$$

By the triangle inequality,

$$\begin{aligned}\sqrt{\int_{\mathfrak{M}_{8,r_0}} |S_{1,\eta_+}(\alpha,x)|^2\, d\alpha} &= \sqrt{\int_{\mathfrak{M}_{8,r_0}} |S_{\eta_+}(\alpha,x) - S_{2,\eta_+}(\alpha,x)|^2\, d\alpha}\\ &\geq \sqrt{\int_{\mathfrak{M}_{8,r_0}} |S_{\eta_+}(\alpha,x)|^2\, d\alpha} - \sqrt{\int_{\mathfrak{M}_{8,r_0}} |S_{2,\eta_+}(\alpha,x)|^2\, d\alpha}\\ &\geq \sqrt{\int_{\mathfrak{M}_{8,r_0}} |S_{\eta_+}(\alpha,x)|^2\, d\alpha} - \sqrt{\int_{\mathbb{R}/\mathbb{Z}} |S_{2,\eta_+}(\alpha,x)|^2\, d\alpha}.\end{aligned}$$

As we already showed,

$$\int_{\mathbb{R}/\mathbb{Z}} |S_{2,\eta_+}(\alpha,x)|^2\, d\alpha = \sum_{\substack{n \text{ non-prime}\\ \text{or } n\leq\sqrt{x}}} \Lambda(n)^2\eta_+(n/x)^2 \leq E.$$

Thus,

$$I_0 \cdot S \geq (\sqrt{J} - \sqrt{E})^2,$$

and so we are done. □

We now should estimate the integral in (7.45). It is easy to see that

$$\int_{r_0}^{\infty}\frac{1}{r^{3/2}}dr = \frac{2}{r_0^{1/2}},\quad \int_{r_0}^{\infty}\frac{\log r}{r^2}dr = \frac{\log er_0}{r_0},\quad \int_{r_0}^{\infty}\frac{1}{r^2}dr = \frac{1}{r_0},$$
$$\int_{r_0}^{r_1}\frac{1}{r}dr = \log\frac{r^+}{r_0},\quad \int_{r_0}^{\infty}\frac{\log r}{r^{3/2}}dr = \frac{2\log e^2 r_0}{\sqrt{r_0}},\quad \int_{r_0}^{\infty}\frac{\log 2r}{r^{3/2}}dr = \frac{2\log 2e^2 r_0}{\sqrt{r_0}},$$
$$\int_{r_0}^{\infty}\frac{(\log 2r)^2}{r^{3/2}}dr = \frac{2P_2(\log 2r_0)}{\sqrt{r_0}},\quad \int_{r_0}^{\infty}\frac{(\log 2r)^3}{r^{3/2}}dr = \frac{2P_3(\log 2r_0)}{r_0^{1/2}}, \tag{7.54}$$

where

$$P_2(t) = t^2 + 4t + 8,\qquad P_3(t) = t^3 + 6t^2 + 24t + 48. \tag{7.55}$$

We must also estimate the integrals

$$\int_{r_0}^{r_1}\frac{\sqrt{F(r)}}{r^{3/2}}dr,\quad \int_{r_0}^{r_1}\frac{F(r)}{r^2}dr,\quad \int_{r_0}^{r_1}\frac{F(r)\log r}{r^2}dr,\quad \int_{r_0}^{r_1}\frac{F(r)}{r^{3/2}}dr, \tag{7.56}$$

Clearly, $\mathcal{F}(r) - e^\gamma \log\log r = 2.50637/\log\log r$ is decreasing on r. Hence, for $r \geq 10^5$,

$$\mathcal{F}(r) \leq e^\gamma \log\log r + c_\gamma,$$

where $c_\gamma = 1.025742$. Let $F(t) = e^\gamma \log t + c_\gamma$. Then $F''(t) = -e^\gamma/t^2 < 0$. Hence

$$\frac{d^2\sqrt{F(t)}}{dt^2} = \frac{F''(t)}{2\sqrt{F(t)}} - \frac{(F'(t))^2}{4(F(t))^{3/2}} < 0$$

for all $t > 0$. In other words, $\sqrt{F(t)}$ is convex-down, and so we can bound $\sqrt{F(t)}$ from above by $\sqrt{F(t_0)} + \sqrt{F}'(t_0) \cdot (t - t_0)$, for any $t \geq t_0 > 0$. Hence, for $r \geq r_0 \geq 10^5$,

$$\begin{aligned}\sqrt{\mathcal{F}(r)} \leq \sqrt{F(\log r)} &\leq \sqrt{F(\log r_0)} + \frac{d\sqrt{F(t)}}{dt}|_{t=\log r_0} \cdot \log\frac{r}{r_0}\\ &= \sqrt{F(\log r_0)} + \frac{e^\gamma}{\sqrt{F(\log r_0)}} \cdot \frac{\log\frac{r}{r_0}}{2\log r_0}.\end{aligned}$$

Thus, by (7.54),

(7.57)

$$\begin{aligned}\int_{r_0}^\infty \frac{\sqrt{\mathcal{F}(r)}}{r^{3/2}} dr &\leq \sqrt{F(\log r_0)}\left(2 - \frac{e^\gamma}{F(\log r_0)}\right)\frac{1}{\sqrt{r_0}} + \frac{e^\gamma}{\sqrt{F(\log r_0)}\log r_0}\frac{\log e^2 r_0}{\sqrt{r_0}}\\ &= \frac{2\sqrt{F(\log r_0)}}{\sqrt{r_0}}\left(1 - \frac{e^\gamma}{F(\log r_0)\log r_0}\right).\end{aligned}$$

The other integrals in (7.56) are easier. Just as in (7.57), we extend the range of integration to $[r_0, \infty]$. Using (7.54), we obtain

$$\begin{aligned}\int_{r_0}^\infty \frac{\mathcal{F}(r)}{r^2} dr \leq \int_{r_0}^\infty \frac{F(\log r)}{r^2} dr &= e^\gamma\left(\frac{\log\log r_0}{r_0} + E_1(\log r_0)\right) + \frac{c_\gamma}{r_0},\\ \int_{r_0}^\infty \frac{\mathcal{F}(r)\log r}{r^2} dr &\leq e^\gamma\left(\frac{(1+\log r_0)\log\log r_0 + 1}{r_0} + E_1(\log r_0)\right) + \frac{c_\gamma \log e r_0}{r_0},\end{aligned}$$

where E_1 is the *exponential integral*

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt.$$

By [OLBC10, (6.8.2)],

$$\frac{1}{r(\log r + 1)} \leq E_1(\log r) \leq \frac{1}{r\log r}.$$

Hence

$$\begin{aligned}\int_{r_0}^\infty \frac{\mathcal{F}(r)}{r^2} dr &\leq \frac{e^\gamma(\log\log r_0 + 1/\log r_0) + c_\gamma}{r_0},\\ \int_{r_0}^\infty \frac{\mathcal{F}(r)\log r}{r^2} dr &\leq \frac{e^\gamma\left(\log\log r_0 + \frac{1}{\log r_0}\right) + c_\gamma}{r_0} \cdot \log e r_0.\end{aligned}$$

Finally,

$$\int_{r_0}^{r_1} \frac{F(r)}{r^{3/2}} \leq -e^\gamma \left(\frac{2\log\log r}{\sqrt{r}} + 2E_1\left(\frac{\log r}{2}\right)\right)|_{r_0}^{r_1} + \left(\frac{2c_\gamma}{\sqrt{r_0}} - \frac{2c_\gamma}{\sqrt{r_1}}\right)$$
$$\leq 2e^\gamma \left(\frac{\log\log r_0}{\sqrt{r_0}} - \frac{\log\log r_1}{\sqrt{r_1}}\right) + 2c_\gamma\left(\frac{1}{\sqrt{r_0}} - \frac{1}{\sqrt{r_1}}\right)$$
$$+ 4e^\gamma\left(\frac{1}{\sqrt{r_0}\log r_0} - \frac{1}{\sqrt{r_1}(\log r_1 + 2)}\right).$$

It is time to estimate

$$\int_{r_0}^{r_1} \frac{R_{y,2r}\log 2r\sqrt{F(r)}}{r^{3/2}} dr, \tag{7.58}$$

where $z = y$ or $z = y/\log y$ (and $y = x/\varkappa$, as before). By Cauchy-Schwarz, (7.58) is at most

$$\sqrt{\int_{r_0}^{r_1} \frac{(R_{z,2r}\log 2r)^2}{r^{3/2}} dr} \cdot \sqrt{\int_{r_0}^{r_1} \frac{F(r)}{r^{3/2}} dr}.$$

We have already bounded the second integral. Let us look at $R_{z,t}$ (defined in (4.40)). We can write $R_{z,t} = 0.27125 R^\circ_{z,t} + 0.41415$, where

$$R^\circ_{z,t} = \log\left(1 + \frac{\log 4t}{2\log\frac{9z^{1/3}}{2.004t}}\right). \tag{7.59}$$

Clearly,

$$R^\circ_{z,e^t/4} = \log\left(1 + \frac{t/2}{\log\frac{36z^{1/3}}{2.004} - t}\right).$$

Now, for $f(t) = \log(c + at/(b-t))$ and $t \in [0, b)$,

$$f'(t) = \frac{ab}{\left(c + \frac{at}{b-t}\right)(b-t)^2}, \qquad f''(t) = \frac{-ab((a-2c)(b-2t) - 2ct)}{\left(c+\frac{at}{b-t}\right)^2 (b-t)^4}.$$

In our case, $a = 1/2$, $c = 1$ and $b = \log 36z^{1/3} - \log(2.004) > 0$. Hence, for $t < b$,

$$-ab((a-2c)(b-2t) - 2ct) = \frac{b}{2}\left(2t + \frac{3}{2}(b-2t)\right) = \frac{b}{2}\left(\frac{3}{2}b - t\right) > 0,$$

and so $f''(t) > 0$. In other words, $t \to R^\circ_{z,e^t/4}$ is convex-up for $t < b$, i.e., for $e^t/4 < 9z^{1/3}/2.004$. It is easy to check that, since we are assuming $y \geq 10^{24}$,

$$2r_1 = \frac{4}{3} y^{\frac{0.55}{2}} < \frac{9}{2.004}\left(\frac{y}{\log y}\right)^{1/3} \leq \frac{9z^{1/3}}{2.004}.$$

We conclude that $r \to R^\circ_{z,2r}$ is convex-up on $\log 8r$ for $r \leq r_1$, and hence so is $r \to R_{z,r}$. Thus, for $r \in [r_0, r_1]$,

$$R^2_{z,2r} \leq R^2_{z,2r_0} \cdot \frac{\log r_1/r}{\log r_1/r_0} + R^2_{z,2r_1} \cdot \frac{\log r/r_0}{\log r_1/r_0}. \tag{7.60}$$

Therefore, by (7.54),

$$\int_{r_0}^{r_1} \frac{(R_{z,2r}\log 2r)^2}{r^{3/2}}dr \leq \int_{r_0}^{r_1}\left(R_{z,2r_0}^2\frac{\log r_1/r}{\log r_1/r_0} + R_{z,2r_1}^2\frac{\log r/r_0}{\log r_1/r_0}\right)(\log 2r)^2\frac{dr}{r^{3/2}}$$
$$=\frac{2R_{z,2r_0}^2}{\log\frac{r_1}{r_0}}\left(\left(\frac{P_2(\log 2r_0)}{\sqrt{r_0}} - \frac{P_2(\log 2r_1)}{\sqrt{r_1}}\right)\log 2r_1 - \left(\frac{P_3(\log 2r_0)}{\sqrt{r_0}} - \frac{P_3(\log 2r_1)}{\sqrt{r_1}}\right)\right)$$
$$+\frac{2R_{z,2r_1}^2}{\log\frac{r_1}{r_0}}\left(\left(\frac{P_3(\log 2r_0)}{\sqrt{r_0}} - \frac{P_3(\log 2r_1)}{\sqrt{r_1}}\right) - \left(\frac{P_2(\log 2r_0)}{\sqrt{r_0}} - \frac{P_2(\log 2r_1)}{\sqrt{r_1}}\right)\log 2r_0\right)$$
$$= 2\left(R_{z,2r_0}^2 - \frac{\log 2r_0}{\log\frac{r_1}{r_0}}(R_{z,2r_1}^2 - R_{x,2r_0}^2)\right)\cdot\left(\frac{P_2(\log 2r_0)}{\sqrt{r_0}} - \frac{P_2(\log 2r_1)}{\sqrt{r_1}}\right)$$
$$+ 2\frac{R_{z,2r_1}^2 - R_{z,2r_0}^2}{\log\frac{r_1}{r_0}}\left(\frac{P_3(\log 2r_0)}{\sqrt{r_0}} - \frac{P_3(\log 2r_1)}{\sqrt{r_1}}\right)$$
$$= 2R_{z,2r_0}^2\cdot\left(\frac{P_2(\log 2r_0)}{\sqrt{r_0}} - \frac{P_2(\log 2r_1)}{\sqrt{r_1}}\right)$$
$$+ 2\frac{R_{z,2r_1}^2 - R_{z,2r_0}^2}{\log\frac{r_1}{r_0}}\left(\frac{P_2^-(\log 2r_0)}{\sqrt{r_0}} - \frac{P_3(\log 2r_1) - (\log 2r_0)P_2(\log 2r_1)}{\sqrt{r_1}}\right),$$

where $P_2(t)$ and $P_3(t)$ are as in (7.55), and $P_2^-(t) = P_3(t) - tP_2(t) = 2t^2+16t+48$.

Putting all terms together, we conclude that

$$\int_{r_0}^{r_1}\frac{g(r)}{r}dr \leq f_0(r_0,y) + f_1(r_0) + f_2(r_0,y), \tag{7.61}$$

where

$$\begin{aligned} f_0(r_0,y) &= \left((1-c_\varphi)\sqrt{I_{0,r_0,r_1,y}} + c_\varphi\sqrt{I_{0,r_0,r_1,\frac{y}{\log y}}}\right)\sqrt{\frac{2}{\sqrt{r_0}}I_{1,r_0}} \\ f_1(r_0) &= \frac{\sqrt{F(\log r_0)}}{\sqrt{2r_0}}\left(1 - \frac{e^\gamma}{F(\log r_0)\log r_0}\right) + \frac{5}{\sqrt{2r_0}} \\ &\quad + \frac{1}{r_0}\left(\left(\frac{13}{4}\log er_0 + 10.102\right)J_{r_0} + \frac{80}{9}\log er_0 + 23.433\right) \\ f_2(r_0,y) &= 3.2\frac{(\log y)^{1/6}}{y^{1/6}}\log\frac{r_1}{r_0}, \end{aligned} \tag{7.62}$$

where $F(t) = e^\gamma\log t + c_\gamma$, $c_\gamma = 1.025742$, $y = x/\varkappa$ (as usual),

(7.63)

$$I_{0,r_0,r_1,z} = R_{z,2r_0}^2\cdot\left(\frac{P_2(\log 2r_0)}{\sqrt{r_0}} - \frac{P_2(\log 2r_1)}{\sqrt{r_1}}\right)$$
$$+\frac{R_{z,2r_1}^2 - R_{z,2r_0}^2}{\log\frac{r_1}{r_0}}\left(\frac{P_2^-(\log 2r_0)}{\sqrt{r_0}} - \frac{P_3(\log 2r_1) - (\log 2r_0)P_2(\log 2r_1)}{\sqrt{r_1}}\right)$$
$$J_r = F(\log r) + \frac{e^\gamma}{\log r},\qquad I_{1,r} = F(\log r) + \frac{2e^\gamma}{\log r},\qquad c_\varphi = \frac{C_{\varphi,2,\log y}/|\varphi|_1}{\log\log y}$$

and $C_{\varphi,2,K}$ is as in (4.47).

8. Conclusion

We now need to gather all results, using the smoothing functions

$$\eta_+ = h_{200}(t)\eta_\heartsuit(t)$$

(as in (4.7), with $H = 200$) and

$$\eta_* = (\eta_2 *_M \varphi)(\varkappa t)$$

(as in Thm. 7.10, with $\kappa \le 1750$ to be set soon). We define $\varphi(t) = t^2 e^{-t^2/2}$. Just like before, $h_H = h_{200}$ is as in (4.10), $\eta_\heartsuit(t) = \eta_{\heartsuit,1}(t) = e^{-t^2/2}$, and $\eta_2 = \eta_1 *_M \eta_1$, where $\eta_1 = 2 \cdot I_{[-1/2,1/2]}$.

We fix a value for r_0, namely, $r_0 = 150000$. Our results will have to be valid for any $x \ge x_1$, where x_1 is fixed. We set $x_1 = 4.5 \cdot 10^{29}$, since we want a result valid for $N \ge 10^{30}$, and, as was discussed in (4.1), we will work with x_1 slightly smaller than $N/2$. (We will later see that $N \ge 10^{30}$ implies the stronger lower bound $x \ge 4.9 \cdot 10^{29}$.)

8.1. **The ℓ_2 norm over the major arcs.** We apply Lemma 3.1 with $\eta = \eta_+$ and $\eta_\circ$ as in (4.3). Let us first work out the error terms. Recall that $\delta_0 = 8$. We use the bound on E_{η_+,r,δ_0} given in (6.63), and the bound on $ET_{\eta_+,\delta_0 r/2}$ in (6.64):

$$E_{\eta_+,r,\delta_0} \le 2.139 \cdot 10^{-8}, \qquad ET_{\eta_+,\delta_0 r/2} \le 5.122 \cdot 10^{-9}.$$

We also need to bound a few norms: by (4.17), (4.18), (4.26), (4.31) and (4.30),

$$\begin{aligned} |\eta_+|_1 &\le 0.996505, \\ |\eta_+|_2 &\le 0.80013 + \frac{547.5562}{200^{7/2}} \le 0.80014, \\ |\eta_+|_\infty &\le 1 + 2.21526 \cdot \frac{1 + \frac{4}{\pi}\log 200}{200} \le 1.0858. \end{aligned} \tag{8.1}$$

By (3.12),

$$\begin{aligned} S_{\eta_+}(0,x) &= \widehat{\eta_+}(0) \cdot x + O^*\left(\mathrm{err}_{\eta_+,\chi_T}(0,x)\right) \cdot x \\ &\le (|\eta_+|_1 + ET_{\eta_+,\delta_0 r/2})x \le 0.99651x. \end{aligned}$$

Hence, we can bound $K_{r,2}$ in (3.27):

$$\begin{aligned} K_{r,2} &= (1 + \sqrt{300000})(\log x)^2 \cdot 1.14146 \\ &\quad \cdot (2 \cdot 0.99651 + (1 + \sqrt{300000})(\log x)^2 \cdot 1.14146/x) \\ &= 1248.32(\log x)^2 \le 2.7 \cdot 10^{-14} x \end{aligned}$$

for $x \ge 10^{20}$. We also have

$$5.19\delta_0 r \left(ET_{\eta_+,\frac{\delta_0 r}{2}} \cdot \left(|\eta|_1 + \frac{ET_{\eta_+,\frac{\delta_0 r}{2}}}{2} \right) \right) \le 0.031789$$

and

$$\delta_0 r \left(\left(2 + \frac{3\log r}{2} \right) \cdot E^2_{\eta_+,r,\delta_0} + (\log 2e^2 r) K_{r,2} \right) \le 4.844 \cdot 10^{-7}.$$

We recall from (4.18) and (4.14) that

$$0.8001287 \le |\eta_\circ|_2 \le 0.8001288 \tag{8.2}$$

and

$$|\eta_+ - \eta_\circ|_2 \le \frac{547.5562}{H^{7/2}} \le 4.84 \cdot 10^{-6}. \tag{8.3}$$

Symbolic integration gives

$$|\eta_\circ'|_2^2 = 2.7375292\ldots \tag{8.4}$$

We bound $|\eta_\circ^{(3)}|_1$ using the fact that (as we can tell by taking derivatives) $\eta_\circ^{(2)}(t)$ increases from 0 at $t=0$ to a maximum within $[0,1/2]$, and then decreases to $\eta_\circ^{(2)}(1) = -7$, only to increase to a maximum within $[3/2,2]$ (equal to that within $[0,1/2]$) and then decrease to 0 at $t=2$:

$$\begin{aligned}|\eta_\circ^{(3)}|_1 &= 2 \max_{t\in[0,1/2]} \eta_\circ^{(2)}(t) - 2\eta_\circ^{(2)}(1) + 2 \max_{t\in[3/2,2]} \eta_\circ^{(2)}(t)\\ &= 4 \max_{t\in[0,1/2]} \eta_\circ^{(2)}(t) + 14 \leq 4\cdot 4.6255653 + 14 \leq 32.503,\end{aligned} \tag{8.5}$$

where we compute the maximum by the bisection method with 30 iterations (using interval arithmetic, as always).

We evaluate explicitly

$$\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} = 13.597558346\ldots$$

Looking at (3.29) and $L_{r,\delta_0} \leq (\log r + 1.7)|\eta_+|_2^2$,, we conclude that

$$\begin{aligned}L_{r,\delta_0} &\leq 13.597558347 \cdot 0.8001287^2 \leq 8.70524,\\ L_{r,\delta_0} &\geq 13.597558348 \cdot 0.8001288^2 + O^*((\log r + 1.7)\cdot 4.84\cdot 10^{-6})\\ &\quad + O^*\left(1.341\cdot 10^{-5}\right)\cdot\left(0.64787 + \frac{\log r}{4r} + \frac{0.425}{r}\right) \geq 8.70516.\end{aligned}$$

Lemma 3.1 thus gives us that

$$\begin{aligned}\int_{\mathfrak{M}_{8,r_0}} \left|S_{\eta_+}(\alpha,x)\right|^2 d\alpha &= (8.7052 + O^*(0.0004))x + O^*(0.03179)x\\ &= (8.7052 + O^*(0.0322))x.\end{aligned} \tag{8.6}$$

8.2. **The total major-arcs contribution.** First of all, we must bound from below

$$C_0 = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\cdot\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right). \tag{8.7}$$

The only prime that we know does not divide N is 2. Thus, we use the bound

$$C_0 \geq 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \geq 1.3203236.$$

The other main constant is $C_{\eta_\circ,\eta_*}$, which we defined in (3.37) and already started to estimate in (4.6):

$$C_{\eta_\circ,\eta_*} = |\eta_\circ|_2^2 \int_0^2 \eta_*(\rho)d\rho + 2.71|\eta_\circ'|_2^2 \cdot O^*\left(\int_0^{\frac{N}{x}} ((2-N/x)+\rho)^2 \eta_*(\rho)d\rho\right) \tag{8.8}$$

provided that $N \geq 2x$. Recall that $\eta_* = (\eta_2 *_M \varphi)(\varkappa t)$, where $\varphi(t) = t^2 e^{-t^2/2}$. Therefore,

$$\int_0^{N/x} \eta_*(\rho) d\rho = \int_0^{N/x} (\eta_2 * \varphi)(\varkappa\rho) d\rho = \int_{1/4}^1 \eta_2(w) \int_0^{N/x} \varphi\left(\frac{\varkappa\rho}{w}\right) d\rho \frac{dw}{w}$$
$$= \frac{|\eta_2|_1 |\varphi|_1}{\varkappa} - \frac{1}{\varkappa}\int_{1/4}^1 \eta_2(w) \int_{\varkappa N/xw}^{\infty} \varphi(\rho) d\rho dw.$$

Now

$$\int_y^\infty \varphi(\rho) = y e^{-y^2/2} + \sqrt{2}\int_{y/\sqrt{2}}^\infty e^{-t^2} dt < \left(y + \frac{2}{y}\right) e^{-y^2/2}$$

by [OLBC10, (7.8.3)]. Hence

$$\int_{\varkappa N/xw}^\infty \varphi(\rho) d\rho \leq \int_{2\varkappa}^\infty \varphi(\rho) d\rho < \left(2\varkappa + \frac{1}{\varkappa}\right) e^{-2\varkappa^2}$$

and so, since $|\eta_2|_1 = 1$,

$$\int_0^{N/x} \eta_*(\rho) d\rho \geq \frac{|\varphi|_1}{\varkappa} - \int_{1/4}^1 \eta_2(w) dw \cdot \left(2 + \frac{1}{\varkappa^2}\right) e^{-2\varkappa^2}$$
$$\geq |\varphi|_1 - \left(2 + \frac{1}{\varkappa^2}\right) e^{-2\varkappa^2}. \tag{8.9}$$

Let us now focus on the second integral in (8.8). Write $N/x = 2 + c_1/\varkappa$. Then the integral equals

$$\int_0^{2+c_1/\varkappa} (-c_1/\varkappa + \rho)^2 \eta_*(\rho) d\rho \leq \frac{1}{\varkappa^3}\int_0^\infty (u - c_1)^2 \; (\eta_2 *_M \varphi)(u) \; du$$
$$= \frac{1}{\varkappa^3}\int_{1/4}^1 \eta_2(w) \int_0^\infty (vw - c_1)^2 \varphi(v) dv dw$$
$$= \frac{1}{\varkappa^3}\int_{1/4}^1 \eta_2(w) \left(3\sqrt{\frac{\pi}{2}} w^2 - 2 \cdot 2c_1 w + c_1^2 \sqrt{\frac{\pi}{2}}\right) dw$$
$$= \frac{1}{\varkappa^3}\left(\frac{49}{48}\sqrt{\frac{\pi}{2}} - \frac{9}{4} c_1 + \sqrt{\frac{\pi}{2}} c_1^2\right).$$

It is thus best to choose $c_1 = (9/4)/\sqrt{2\pi} = 0.89762\ldots$. Looking up $|\eta_\circ'|_2^2$ in (8.4), we obtain

$$2.71 |\eta_\circ'|_2^2 \cdot \int_0^{\frac{N}{x}} ((2 - N/x) + \rho)^2 \eta_*(\rho) d\rho$$
$$\leq 7.413 \cdot \frac{1}{\varkappa^3}\left(\frac{49}{48}\sqrt{\frac{\pi}{2}} - \frac{(9/4)^2}{2\sqrt{2\pi}}\right) \leq \frac{1.9986}{\varkappa^3}.$$

We conclude that

$$C_{\eta_\circ, \eta_*} \geq \frac{1}{\varkappa} |\varphi|_1 |\eta_\circ|_2^2 - |\eta_\circ|_2^2 \left(2 + \frac{1}{\varkappa^2}\right) e^{-2\varkappa^2} - \frac{1.9986}{\varkappa^3}.$$

Setting

$$\varkappa = 49$$

and using (4.18), we obtain

$$C_{\eta_\circ, \eta_*} \geq \frac{1}{\varkappa}(|\varphi|_1 |\eta_\circ|_2^2 - 0.000833). \tag{8.10}$$

Here it is useful to note that $|\varphi|_1 = \sqrt{\frac{\pi}{2}}$, and so, by (4.18), $|\varphi|_1|\eta_\circ|_2^2 = 0.80237\ldots$.

We have finally chosen x in terms of N:

$$x = \frac{N}{2+\frac{c_1}{\varkappa}} = \frac{N}{2+\frac{49/36}{\sqrt{2\pi}}\frac{1}{49}} = \frac{N}{2+\frac{1}{36\sqrt{2\pi}}} = 0.49724\ldots \cdot N. \tag{8.11}$$

Thus, we see that, since we are assuming $N \geq 10^{30}$, we in fact have $x \geq 4.9724\ldots \cdot 10^{29}$, and so, in particular,

$$x \geq 4.9 \cdot 10^{29}, \quad \frac{x}{\varkappa} \geq 10^{28}. \tag{8.12}$$

Let us continue with our determination of the major-arcs total. We should compute the quantities in (3.38). We already have bounds for E_{η_+,r,δ_0}, A_{η_+}, L_{η,r,δ_0} and $K_{r,2}$. From (6.63), we have

$$E_{\eta_*,r,8} \leq \frac{3.061\cdot 10^{-8}}{\varkappa}, \tag{8.13}$$

where the factor of $\varkappa$ comes from the scaling in $\eta_*(t) = (\eta_2 *_M \varphi)(\varkappa t)$ (which in effect divides x by $\varkappa$). It remains only to bound the more harmless terms of type $Z_{\eta,2}$ and LS_η.

By (3.12),

$$\begin{aligned} Z_{\eta_+^2,2} &= \frac{1}{x}\sum_n \Lambda^2(n)\eta_+^2(n/x) \leq \frac{1}{x}\sum_n \Lambda(n)\eta_+(n/x)\cdot(\eta_+(n/x)\log n) \\ &\leq \frac{1}{x}\sum_n \Lambda(n)\eta_+(n/x)\cdot(|\eta_+(t)\cdot\log^+(t/2)|_\infty + |\eta_+|_\infty \log 2x) \\ &\leq (|\eta_+|_1 + |\operatorname{err}_{\eta_+,\chi_T}(0,x)|)\cdot(|\eta_+(t)\cdot\log^+(t/2)|_\infty + |\eta_+|_\infty\log 2x), \end{aligned} \tag{8.14}$$

where $\log^+(t) := \max(0,\log t)$. Proceeding as in (4.27), we see that

$$\begin{aligned} |\eta_+(t)\cdot\log^+(t/2)|_\infty &\leq |\eta_\circ(t)\cdot\log^+(t/2)|_\infty + |h-h_H|_\infty\cdot|\eta_\heartsuit(t)\cdot\log^+(t/2)|_\infty \\ &= |h-h_H|_\infty\cdot|\eta_\heartsuit(t)\cdot\log^+(t/2)|_\infty. \end{aligned}$$

Since

$$|\eta_\heartsuit(t)\cdot\log^+(t/2)|_\infty \leq \max_{t\geq 2}|\eta_\heartsuit(t)(t/2-1)| \leq 0.012$$

and $|h-h_H|_\infty < 0.1415$ (by (4.29) and (4.30)), we see that $|\eta_+(t)\cdot\log^+(t/2)|_\infty < 0.0017$. We also have the bounds on $|\eta_+|_1$ and $|\eta_+|_\infty$ in (8.1) and the bound $|\operatorname{err}_{\eta_+,\chi_T}(0,x)| \leq ET_{\eta_+,\delta_0 r/2} \leq 5.122\cdot 10^{-9}$ from (6.64). Hence

$$Z_{\eta_+^2,2} \leq 0.99651\cdot(0.0017 + 1.0858\log 2x) \leq 0.002 + 1.09\log x.$$

Similarly,

$$\begin{aligned} Z_{\eta_*^2,2} &= \frac{1}{x}\sum_n \Lambda^2(n)\eta_*^2(n/x) \leq \frac{1}{x}\sum_n \Lambda(n)\eta_*(n/x)\cdot(\eta_*(n/x)\log n) \\ &\leq (|\eta_*|_1 + |\operatorname{err}_{\eta_*,\chi_T}(0,x)|)\cdot(|\eta_*(t)\cdot\log^+(\varkappa t)|_\infty + |\eta_*|_\infty\log(x/\varkappa)). \end{aligned}$$

Clearly,

$$|\eta_*|_\infty = |\eta_2 *_M \varphi|_\infty \leq \left|\frac{\eta_2(t)}{t}\right|_1 |\varphi|_\infty \leq 1.92182\cdot\frac{2}{e} \leq 1.414. \tag{8.15}$$

and, since $\log^+$ is non-decreasing and η_2 is supported on a subset of $[0,1]$,

$$|\eta_*(t)\cdot\log^+(\varkappa t)|_\infty = |(\eta_2 *_M \varphi)\cdot\log^+|_\infty \leq |\eta_2 *_M (\varphi\cdot\log^+)|_\infty$$
$$\leq \left|\frac{\eta_2(t)}{t}\right|_1 \cdot |\varphi\cdot\log^+|_\infty \leq 1.92182\cdot 0.762313 \leq 1.46503$$

where we bound $|\varphi\cdot\log^+|_\infty$ by the bisection method with 25 iterations. We already know that

$$|\eta_*|_1 = \frac{|\eta_2|_1|\varphi|_1}{\varkappa} = \frac{|\varphi|_1}{\varkappa} = \frac{\sqrt{\pi/2}}{\varkappa}. \tag{8.16}$$

We conclude that

(8.17)

$$Z_{\eta_*^2,2} \leq (\sqrt{\pi/2}/49 + 3.061\cdot 10^{-8})(1.46503 + (2/e)\log(x/49)) \leq 0.0189\log x.$$

We have bounds for $|\eta_*|_\infty$ and $|\eta_+|_\infty$. We can also bound

$$|\eta_*\cdot t|_\infty = \kappa|(\eta_2 *_M \varphi)\cdot t|_\infty \leq \kappa|\eta_2|_1\cdot|\varphi\cdot t|_\infty \leq 3^{3/2}e^{-3/2}$$

and

$$\begin{aligned}|\eta_+(t)\cdot t|_\infty &\leq |\eta_\circ(t)\cdot t|_\infty + |h-h_H|_\infty\cdot|\eta_\heartsuit(t)\cdot t|_\infty\\ &\leq 1.0648 + 0.1415e^{-1/2} \leq 1.1507,\end{aligned} \tag{8.18}$$

where we bound $|\eta_\heartsuit(t)\cdot t|_\infty \leq 1.0648$ by the bisection method (20 iterations).

We can now bound $LS_\eta(x,r)$ for $\eta = \eta_*, \eta_+$:

$$\begin{aligned}LS_\eta(x,r) &= \log r\cdot\max_{p\leq r}\sum_{\alpha\geq 1}\eta\left(\frac{p^\alpha}{x}\right)\\ &= (\log r)\cdot\max_{p\leq r}\left(\frac{\log x}{\log p}|\eta|_\infty + \sum_{\substack{\alpha\geq 1\\ p^\alpha\geq x}}\frac{|\eta\cdot t|_\infty}{p^\alpha/x}\right)\\ &\leq (\log r)\cdot\max_{p\leq r}\left(\frac{\log x}{\log p}|\eta|_\infty + \frac{|\eta\cdot t|_\infty}{1-1/p}\right)\\ &\leq \frac{(\log r)(\log x)}{\log 2}|\eta|_\infty + 2(\log r)|\eta\cdot t|_\infty,\end{aligned}$$

and so

(8.19)

$$LS_{\eta_*} \leq \left(\frac{2}{e\log 2}\log x + 2\cdot(3/e)^{3/2}\right)\log r \leq (1.0615\log x + 2.3189)\log r,$$

$$LS_{\eta_+} \leq \left(\frac{1.14146}{\log 2}\log x + 2\cdot 1.1507\right)\log r \leq (1.6468\log x + 2.3014)\log r.$$

We can now start to put together all terms in (3.36). Let $\epsilon_0 = |\eta_+ - \eta_\circ|_2/|\eta_\circ|_2$. Then, by (8.3),

$$\epsilon_0|\eta_\circ|_2 \leq |\eta_+ - \eta_\circ|_2 \leq 4.84\cdot 10^{-6}.$$

Thus,

$$2.82643|\eta_\circ|_2^2(2+\epsilon_0)\cdot\epsilon_0 + \frac{4.31004\delta_0|\eta_\circ|_1^2 + 0.0012\frac{|\eta_\circ^{(3)}|_1^2}{\delta_0^5}}{r}$$

is at most

$$2.82643 \cdot 4.84 \cdot 10^{-6} \cdot (2 \cdot 0.80013 + 4.84 \cdot 10^{-6}) + \frac{4.311 \cdot 8 \cdot 0.64021 + 0.0012 \cdot \frac{32.503^2}{8^5}}{150000} \leq 0.0001691$$

by (4.18) and (8.5). Recall that $|\eta_*|_1$ is given by (8.16).

Since $\eta_* = (\eta_2 *_M \varphi)(\varkappa x)$,

$$|\eta_*|_2^2 = \frac{|\eta_2 *_M \varphi|_2^2}{\varkappa} \leq \frac{|\eta_2(t)/\sqrt{t}|_2 \cdot |\varphi|_2}{\varkappa} = \frac{\sqrt{\frac{3}{8}\sqrt{\pi} \cdot \frac{32}{3}(\log 2)^3}}{\varkappa} \leq \frac{1.53659}{\varkappa}.$$

The second line of (3.36) is thus at most x^2 times

$$\frac{3.061 \cdot 10^{-8}}{\varkappa} \cdot 8.739 + 2.139 \cdot 10^{-8} \cdot 1.6812(\sqrt{8.739} + 1.6812 \cdot 0.80014)\sqrt{\frac{1.53659}{\varkappa}} \leq \frac{1.6097 \cdot 10^{-6}}{\varkappa}.$$

where we are using the bound $A_{\eta_+} \leq 8.739$ we obtained in (8.6). (We are also using the bounds on norms in (8.1).)

Using the bounds (8.14), (8.17) and (8.19), we see that the third line of (3.36) is at most

$$\begin{aligned}&2 \cdot (0.0189 \log x \cdot (1.0615 \log x + 2.3189) \log 150000) \cdot x\\ &+ 4\sqrt{(0.002 + 1.14 \log x) \cdot 0.0189 \log x}(1.6468 \log x + 2.3014)(\log 150000)x\\ &\leq 13(\log x)^2 x,\end{aligned}$$

where we just use the very weak assumption $x \geq 10^{10}$, though we can by now assume (8.12)).

We conclude that, for $r = 150000$, the integral over the major arcs

$$\int_{\mathfrak{M}_{8,r}} S_{\eta_+}(\alpha, x)^2 S_{\eta_*}(\alpha, x) e(-N\alpha) d\alpha$$

is

(8.20)

$$\begin{aligned}&C_0 \cdot C_{\eta_0,\eta_*} x^2 + O^*\left(0.0001691 \cdot \frac{\sqrt{\pi/2}}{\varkappa} x^2 + \frac{1.6097 \cdot 10^{-6}}{\varkappa} x^2 + 14(\log x)^2 x\right)\\ &= C_0 \cdot C_{\eta_0,\eta_*} x^2 + O^*\left(\frac{0.0002136 x^2}{\varkappa}\right) = C_0 \cdot C_{\eta_0,\eta_*} x^2 + O^*\left(4.359 \cdot 10^{-6} x^2\right),\end{aligned}$$

where C_0 and C_{η_0,η_*} are as in (3.37). Notice that $C_0 C_{\eta_0,\eta_*} x^2$ is the expected asymptotic for the integral over all of $\mathbb{R}/\mathbb{Z}$.

Moreover, by (8.9) and (8.10),

$$C_0 \cdot C_{\eta_0,\eta_*} \geq \frac{1.3203236 |\varphi|_1 |\eta_\circ|_2^2}{\varkappa} - \frac{0.000833}{\varkappa} \geq \frac{1.0594003}{\varkappa} - \frac{0.000833}{\varkappa} = \frac{1.05857}{49}.$$

Hence

$$\int_{\mathfrak{M}_{8,r}} S_{\eta_+}(\alpha, x)^2 S_{\eta_*}(\alpha, x) e(-N\alpha) d\alpha \geq \frac{1.05856}{\varkappa} x^2, \tag{8.21}$$

where, as usual, $\varkappa = 49$. This is our total major-arc bound.

8.3. **Minor-arc totals.** We need to estimate the quantities E, S, T, J, M in Theorem 7.10. Let us start by bounding the constants in (7.44). The constants $C_{\eta_+,j}$, $j = 0, 1, 2$, will appear only in the minor term A_2, and so crude bounds on them will do.

By (8.1) and (8.18),

$$\sup_{r\geq t} \eta_+(r) \leq \min\left(1.0858, \frac{1.1507}{t}\right)$$

for all $t \geq 0$. Thus,

$$\begin{aligned} C_{\eta_+,0} &= 0.7131 \int_0^\infty \frac{1}{\sqrt{t}} \left(\sup_{r\geq t} \eta_+(r)\right)^2 dt \\ &\leq 0.7131 \left(\int_0^1 \frac{1.0858^2}{\sqrt{t}} + \int_1^\infty \frac{1.1507^2}{t^{5/2}} dt\right) \leq 2.31092. \end{aligned}$$

Similarly,

$$C_{\eta_+,1} \leq 0.7131 \int_1^\infty \frac{1}{\sqrt{t}} \left(\sup_{r\geq t} \eta_+(r)\right)^2 dt \leq 0.7131 \int_1^\infty \frac{1.1507^2 \log t}{t^{5/2}} dt \leq 0.41965.$$

Immediately from (8.1),

$$C_{\eta_+,2} = 0.51941 |\eta_+|_\infty^2 \leq 0.61235.$$

We get

$$\begin{aligned} E &\leq ((2.488 + 0.677) \log x + (2 \cdot 2.488 + 0.42)) \cdot x^{1/2} \\ &\leq (3.165 \log x + 5.396) \cdot x^{1/2} \leq 3.31 \cdot 10^{-13} \cdot x, \end{aligned} \tag{8.22}$$

where E is defined as in (7.43), and where we are using the assumption $x \geq 4.5 \cdot 10^{29}$ made at the beginning. Using (8.13) and (8.16), we see that

$$S_{\eta_*}(0, x) = (|\eta_*|_1 + O^*(ET_{\eta_*,0}))x = \left(\sqrt{\pi/2} + O^*(3.061 \cdot 10^{-8})\right) \frac{x}{\varkappa}.$$

Hence

$$S_{\eta_*}(0, x) \cdot E \leq 5.84 \cdot 10^{-14} \cdot \frac{x^2}{\varkappa}. \tag{8.23}$$

We can bound

$$S \leq \sum_n \Lambda(n) (\log n) \eta_+^2(n/x) = 0.64022 x \log x - 0.0211 x \tag{8.24}$$

by (6.67). Let us now estimate T. Recall that $\varphi(t) = t^2 e^{-t^2/2}$. Since

$$\int_0^u \varphi(t) dt = \int_0^u t^2 e^{-t^2/2} dt \leq \int_0^u t^2 dt = \frac{u^3}{3},$$

we can bound

$$C_{\varphi,3}\left(\log \frac{x}{\varkappa}\right) = \frac{1.04488}{\sqrt{\pi/2}} \int_0^{\frac{1}{\log x/\varkappa}} t^2 e^{-t^2/2} dt \leq \frac{0.2779}{(\log x/\varkappa)^3}.$$

By (8.6), we already know that $J = (8.7052 + O^*(0.0322))x$. Hence

$$\begin{aligned} (\sqrt{J} - \sqrt{E})^2 &= (\sqrt{(8.7052 + O^*(0.0322))x} - \sqrt{3.302 \cdot 10^{-13} \cdot x})^2 \\ &\geq 8.672999 x, \end{aligned} \tag{8.25}$$

and so

$$\begin{aligned}T &= C_{\varphi,3}\left(\log\frac{x}{\varkappa}\right)\cdot(S-(\sqrt{J}-\sqrt{E})^2)\\ &\leq \frac{0.2779}{(\log x/\varkappa)^3}\cdot((0.6402+O^*(2\cdot 10^{-5}))x\log x - 0.0211x - 8.672999x)\\ &\leq 0.17792\frac{x\log x}{(\log x/\varkappa)^3} - 2.41609\frac{x}{(\log x/\varkappa)^3}\\ &\geq 0.17792\frac{x}{(\log x/\varkappa)^2} - 1.72365\frac{x}{(\log x/\varkappa)^3}.\end{aligned}$$

for $\varkappa = 49$. Since $x/\varkappa \geq 10^{28}$, this implies that

$$T \leq 3.638\cdot 10^{-5}\cdot x. \tag{8.26}$$

It remains to estimate M. Let us first look at $g(r_0)$; here $g = g_{x/\varkappa,\varphi}$, where $g_{x/\varkappa,\varphi}$ is defined as in (4.46). Write $y = x/\varkappa$. We must estimate the constant $C_{\varphi,2,K}$ defined in (4.48):

$$\begin{aligned}C_{\varphi,2,K} &= -\int_{1/K}^1 \varphi(w)\log w dw \leq -\int_0^1 \varphi(w)\log w dw \leq -\int_0^1 w^2 e^{-w^2/2}\log w dw\\ &\leq 0.093426,\end{aligned}$$

where again we use VNODE-LP for rigorous numerical integration. Since $|\varphi|_1 = \sqrt{\pi/2}$, this implies that

$$\frac{C_{\varphi,2}/|\varphi|_1}{\log K} \leq \frac{0.07455}{\log\log y} \tag{8.27}$$

and so

$$R_{y,K,\varphi,t} = \frac{0.07455}{\log\log y}R_{y/K,t} + \left(1-\frac{0.07455}{\log\log y}\right)R_{y,t}. \tag{8.28}$$

Let $t = 2r_0 = 300000$; we recall that $K = \log y$. Recall from (8.12) that $y = x/\varkappa \geq 10^{28}$; thus, $y/K \geq 1.5511\cdot 10^{26}$ and $\log\log y \geq 4.16624$. Going back to the definition of $R_{x,t}$ in (4.40), we see that

$$R_{y,,2r_0} \leq 0.27125\log\left(1+\frac{\log(8\cdot 150000)}{2\log\frac{9\cdot(10^{28})^{1/3}}{2.004\cdot 2\cdot 150000}}\right) + 0.41415 \leq 0.55394, \tag{8.29}$$

(8.30)

$$R_{y/K,2r_0} \leq 0.27125\log\left(1+\frac{\log(8\cdot 150000)}{2\log\frac{9\cdot(1.5511\cdot 10^{26})^{1/3}}{2.004\cdot 2\cdot 150000}}\right) + 0.41415 \leq 0.5703,$$

and so

$$R_{y,K,\varphi,2r_0} \leq \frac{0.07455}{4.16624}0.5703 + \left(1-\frac{0.07455}{4.16624}\right)0.55394 \leq 0.55424.$$

Using

$$F(r) = e^\gamma\log\log r + \frac{2.50637}{\log\log r} \leq 5.42506,$$

we see from (4.40) that

$$L_{r_0} = 5.42506\cdot\left(\log 2^{\frac{7}{4}}150000^{\frac{13}{4}} + \frac{80}{9}\right) + \log 2^{\frac{16}{9}}150000^{\frac{80}{9}} + \frac{111}{5} \leq 394.316.$$

Going back to (4.46), we sum up and obtain that

$$g(r_0) = \frac{(0.55424 \cdot \log 300000 + 0.5)\sqrt{5.42506} + 2.5}{\sqrt{2 \cdot 150000}} + \frac{394.316}{150000} + 3.2\left(\frac{\log y}{y}\right)^{1/6}$$
$$\leq 0.039182.$$

Using again the bound $x \geq 4.9 \cdot 10^{29}$ from (8.12), we obtain

$$\frac{\log(150000+1) + 1.698}{\log\sqrt{x/16}} \cdot S - (\sqrt{J} - \sqrt{E})^2$$
$$\leq \frac{13.6164}{\frac{1}{2}\log x - \log 4} \cdot (0.64022x \log x - 0.0211x) - 8.673x$$
$$\leq 13.6164 \cdot 1.33393x - 8.673x \leq 9.49046x.$$

Therefore,

(8.31)
$$g(r_0) \cdot \left(\frac{\log(150000+1) + 1.698}{\log\sqrt{x/16}} \cdot S - (\sqrt{J} - \sqrt{E})^2\right) \leq 0.039182 \cdot 9.49046x$$
$$\leq 0.37186x.$$

This is one of the main terms.

Let $r_1 = (2/3)y^{0.55/2}$, where, as usual, $y = x/\varkappa$ and $\varkappa = 49$. Then

(8.32)
$$\begin{aligned} R_{y,2r_1} &= 0.27125 \log\left(1 + \frac{\log\left(8 \cdot \frac{2}{3} y^{\frac{0.55}{2}}\right)}{2 \log \frac{9y^{1/3}}{2.004 \cdot 2 \cdot \frac{2}{3} y^{\frac{0.55}{2}}}}\right) + 0.41415 \\ &= 0.27125 \log\left(1 + \frac{\frac{0.55}{2}\log y + \log\frac{16}{3}}{\frac{7}{60}\log y + 2\log\frac{27/8}{1.002}}\right) + 0.41415 \\ &\leq 0.27125 \log\left(1 + \frac{33}{14}\right) + 0.41415 \leq 0.74266. \end{aligned}$$

Similarly, for $K = \log y$ (as usual),

(8.33)
$$\begin{aligned} R_{y/K,2r_1} &= 0.27125 \log\left(1 + \frac{\log\left(8 \cdot \frac{2}{3} y^{\frac{0.55}{2}}\right)}{2 \log \frac{9(y/K)^{1/3}}{2.004 \cdot 2 \cdot \frac{2}{3} y^{\frac{0.55}{2}}}}\right) + 0.41415 \\ &= 0.27125 \log\left(1 + \frac{\frac{0.55}{2}\log y + \log\frac{16}{3}}{\frac{7}{60}\log y - \frac{2}{3}\log\log y + 2\log\frac{27/8}{1.002}}\right) + 0.41415. \end{aligned}$$

Let

$$f(t) = \frac{\frac{0.55}{2}t + \log\frac{16}{3}}{\frac{7}{60}t - \frac{2}{3}\log t + 2\log\frac{27/8}{1.002}} = \frac{33}{14} + \frac{\frac{2}{3}\log t - c}{\frac{7}{60}t - \frac{2}{3}\log t + 2\log\frac{27/8}{1.002}},$$

where $c = (33/14)\log((27/8)/1.002) - \log 16/3$. Then

$$f'(t) = \frac{\frac{2}{3t}\left(\frac{7}{60}t - \frac{2}{3}\log t + 2\log\frac{27/8}{1.002}\right) - \left(\frac{2}{3}\log t - c\right)\left(\frac{7}{60} - \frac{2}{3t}\right)}{\left(\frac{7}{60}t - \frac{2}{3}\log t + 2\log\frac{27/8}{1.002}\right)^2}$$
$$= -\frac{\frac{7}{90}\log t - \frac{7}{60}(c+2/3) + \left(c - 2\log\frac{27/8}{1.002}\right)\frac{2}{3t}}{\left(\frac{7}{60}t - \frac{2}{3}\log t + 2\log\frac{27/8}{1.002}\right)^2}.$$

Hence $f'(t) < 0$ for $t \geq 25$. It follows that $R_{y/K,2r_1}$ (with $K = \log y$ and r_1 varying with y) is decreasing on y for $y \geq e^{25}$, and thus, of course, for $y \geq 10^{28}$. We thus obtain from (8.33) that

$$R_{y/K,2r_1} \leq R_{\frac{10^{28}}{\log 10^{28}},2r_1} \leq 0.76994. \tag{8.34}$$

By (8.28), we conclude that

$$R_{y,K,\varphi,2r_1} \leq \frac{0.07455}{4.16624}\cdot 0.76994 + \left(1 - \frac{0.07455}{4.16624}\right)\cdot 0.74266 \leq 0.74315.$$

Since $r_1 = (2/3)y^{0.55/2}$ and $\digamma(r)$ is increasing for $r \geq 27$, we know that

$$\begin{aligned}\digamma(r_1) \leq \digamma(y^{0.55/2}) &= e^\gamma \log\log y^{0.55/2} + \frac{2.50637}{\log\log y^{0.55/2}}\\ &= e^\gamma\log\log y + \frac{2.50637}{\log\log y - \log\frac{2}{0.55}} - e^\gamma \log\frac{2}{0.55} \leq e^\gamma\log\log y - 1.42763\end{aligned}\tag{8.35}$$

for $y \geq 10^{28}$. Hence, (4.40) gives us that

$$\begin{aligned}L_{r_1} &\leq (e^\gamma\log\log y - 1.42763)\left(\log\frac{2^5}{3^{\frac{13}{4}}}y^{\frac{13}{6}} + \frac{80}{9}\right) + \log\frac{2^{\frac{32}{3}}}{3^{\frac{16}{9}}}y^{\frac{44}{9}} + \frac{111}{5}\\ &\leq 3.859\log y\log\log y + 1.7957\log y + 15.65\log\log y + 15.1\\ &\leq (4.29002\log y + 19.28)\log\log y.\end{aligned}$$

Moreover,

$$\sqrt{\digamma(r_1)} = \sqrt{e^\gamma\log\log y - 1.42763} \leq \sqrt{e^\gamma\log\log y} - \frac{1.42763}{2\sqrt{e^\gamma\log\log y}}$$

and so

$$\begin{aligned}&(0.74315\log\frac{4}{3}y^{\frac{11}{40}} + 0.5)\sqrt{\digamma(r_1)}\\ &\leq (0.2044\log y + 0.7138)\sqrt{e^\gamma\log\log y} - \frac{1.42763\cdot(0.2044\log y + 0.7138)}{2\sqrt{e^\gamma\log\log y}}\\ &\leq (0.2728\log y + 0.9527)\sqrt{\log\log y} - 3.64.\end{aligned}$$

Therefore, by (4.46),

$$g_{y,\varphi}(r_1) \leq \frac{(0.2728\log y + 0.9527)\sqrt{\log\log y} - 1.14}{\sqrt{\frac{4}{3} y^{\frac{11}{40}}}} + \frac{(4.29002\log y + 19.28)\log\log y}{\frac{2}{3} y^{\frac{11}{40}}} + \frac{3.2(\log y)^{1/6}}{y^{1/6}} \leq \frac{0.251\log y\sqrt{\log\log y}}{y^{\frac{11}{80}}}.$$

By (8.24) and $y = x/\varkappa = x/49$, we conclude that

(8.36)

$$0.45 g(r_1) S \leq 0.45 \cdot \frac{0.251\log y\sqrt{\log\log y}}{y^{\frac{11}{80}}} \cdot (0.64022\log x - 0.0211)x \leq \frac{0.11295\log y\sqrt{\log\log y}}{y^{\frac{11}{80}}}(0.64022\log y + 2.471)x \leq 0.09186x.$$

It remains only to bound

$$\frac{2S}{\log\frac{x}{16}} \int_{r_0}^{r_1} \frac{g(r)}{r} dr$$

in the expression (7.45) for M. We will use the bound on the integral given in (7.61). The easiest term to bound there is $f_1(r_0)$, defined in (7.62), since it depends only on r_0: for $r_0 = 150000$,

$$f_1(r_0) = 0.0161322\ldots .$$

It is also not hard to bound $f_2(r_0, x)$, also defined in (7.62):

$$f_2(r_0, y) = 3.2\frac{(\log y)^{1/6}}{y^{1/6}} \log\frac{\frac{2}{3}x^{\frac{11}{40}}}{r_0} \leq 3.2\frac{(\log y)^{1/6}}{y^{1/6}}\left(\frac{11}{40}\log y + 0.6648 - \log r_0\right),$$

and so, since $r_0 = 150000$ and $y \geq 10^{28}$,

$$f_2(r_0, y) \leq 0.000895.$$

Let us now look at the terms $I_{1,r}$, c_φ in (7.63). We already saw in (8.27) that

$$c_\varphi = \frac{C_{\varphi,2}/|\varphi|_1}{\log K} \leq \frac{0.07455}{\log\log y} \leq 0.0179.$$

Since $F(t) = e^\gamma \log t + c_\gamma$ with $c_\gamma = 1.025742$,

$$I_{1,r_0} = F(\log r_0) + \frac{2e^\gamma}{\log r_0} = 5.42938\ldots \tag{8.37}$$

It thus remains only to estimate $I_{0,r_0,r_1,z}$ for $z = y$ and $z = y/K$, where $K = \log y$.

We already know that

$$R_{y,2r_0} \leq 0.55394, \quad R_{y/K,2r_0} \leq 0.5703,$$
$$R_{y,2r_1} \leq 0.74266, \quad R_{y/K,2r_1} \leq 0.76994$$

by (8.29), (8.30), (8.32) and (8.34). We also have the trivial bound $R_{z,t} \geq 0.41415$ valid for any z and t for which $R_{z,t}$ is defined.

Omitting negative terms from (7.63), we easily get the following bound, crude but useful enough:

$$I_{0,r_0,r_1,z} \leq R_{z,2r_0}^2 \cdot \frac{P_2(\log 2r_0)}{\sqrt{r_0}} + \frac{R_{z,2r_1}^2 - 0.41415^2}{\log \frac{r_1}{r_0}} \frac{P_2^-(\log 2r_0)}{\sqrt{r_0}},$$

where $P_2(t) = t^2 + 4t + 8$ and $P_2^-(t) = 2t^2 + 16t + 48$. For $z = y$ and $r_0 = 150000$, this gives

$$\begin{aligned} I_{0,r_0,r_1,y} &\leq 0.55394^2 \cdot \frac{P_2(\log 2r_0)}{\sqrt{r_0}} + \frac{0.74266^2 - 0.41415^2}{\log \frac{2y^{\frac{0.55}{2}}}{3r_0}} \cdot \frac{P_2^-(\log 2r_0)}{\sqrt{r_0}} \\ &\leq 0.17232 + \frac{0.55722}{\frac{11}{40}\log y - \log 225000}; \end{aligned}$$

for $z = y/K$, we proceed in the same way, and obtain

$$I_{0,r_0,r_1,y/K} \leq 0.18265 + \frac{0.61773}{\frac{11}{40}\log y - \log 225000}.$$

This gives us

$$\begin{aligned} &(1-c_\varphi)\sqrt{I_{0,r_0,r_1,y}} + c_\varphi \sqrt{I_{0,r_0,r_1,\frac{y}{\log y}}} \\ &\leq 0.9821 \cdot \sqrt{0.17232 + \frac{0.55722}{\frac{11}{40}\log y - \log 225000}} \\ &+ 0.0179 \sqrt{0.18265 + \frac{0.61773}{\frac{11}{40}\log y - \log 225000}}. \end{aligned} \tag{8.38}$$

In particular, since $y \geq 10^{28}$,

$$(1-c_\varphi)\sqrt{I_{0,r_0,r_1,y}} + c_\varphi \sqrt{I_{0,r_0,r_1,\frac{y}{\log y}}} \leq 0.52514.$$

Therefore,

$$f_0(r_0, y) \leq 0.52514 \cdot \sqrt{\frac{2}{\sqrt{r_0}} 5.42939} \leq 0.087932.$$

By (7.61), we conclude that

$$\int_{r_0}^{r_1} \frac{g(r)}{r} dr \leq 0.087932 + 0.061323 + 0.000895 \leq 0.15015.$$

By (8.24) and $x \geq 4.9 \cdot 10^{29}$,

$$\frac{2S}{\log \frac{x}{16}} \leq \frac{2(0.64022 x \log x - 0.0211 x)}{\log x - \log 16} \leq 1.33393 x.$$

Hence

$$\frac{2S}{\log \frac{x}{16}} \int_{r_0}^{r_1} \frac{g(r)}{r} dr \leq 0.20029 x. \tag{8.39}$$

Putting (8.31), (8.36) and (8.39), we conclude that the quantity M defined in (7.45) is bounded by

$$M \leq 0.37186 x + 0.09186 x + 0.20029 x \leq 0.66401 x. \tag{8.40}$$

Gathering the terms from (8.23), (8.26) and (8.40), we see that Theorem 7.10 states that the minor-arc total

$$Z_{r_0} = \int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_*}(\alpha,x)||S_{\eta_+}(\alpha,x)|^2 d\alpha$$

is bounded by

$$\begin{aligned} Z_{r_0} &\leq \left(\sqrt{\frac{|\varphi|_1 x}{\varkappa}(M+T)} + \sqrt{S_{\eta_*}(0,x)\cdot E}\right)^2 \\ &\leq \left(\sqrt{|\varphi|_1(0.6022 + 3.638\cdot 10^{-5})}\frac{x}{\sqrt{\varkappa}} + \sqrt{5.84\cdot 10^{-14}}\frac{x}{\sqrt{\varkappa}}\right)^2 \\ &\leq 0.83226\frac{x^2}{\varkappa} \end{aligned} \tag{8.41}$$

for $r_0 = 150000$, $x \geq 4.9\cdot 10^{29}$, where we use yet again the fact that $|\varphi|_1 = \sqrt{\pi/2}$. This is our total minor-arc bound.

8.4. **Conclusion: proof of main theorem.** As we have known from the start,

$$\begin{aligned} &\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_+(n_1)\eta_+(n_2)\eta_*(n_3) \\ &\qquad = \int_{\mathbb{R}/\mathbb{Z}} S_{\eta_+}(\alpha,x)^2 S_{\eta_*}(\alpha,x)e(-N\alpha)d\alpha. \end{aligned} \tag{8.42}$$

We have just shown that, assuming $N \geq 10^{30}$, N odd,

$$\begin{aligned} &\int_{\mathbb{R}/\mathbb{Z}} S_{\eta_+}(\alpha,x)^2 S_{\eta_*}(\alpha,x)e(-N\alpha)d\alpha \\ &\quad = \int_{\mathfrak{M}_{8,r_0}} S_{\eta_+}(\alpha,x)^2 S_{\eta_*}(\alpha,x)e(-N\alpha)d\alpha \\ &\quad + O^*\left(\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{M}_{8,r_0}} |S_{\eta_+}(\alpha,x)|^2 |S_{\eta_*}(\alpha,x)|d\alpha\right) \\ &\quad \geq 1.05856\frac{x^2}{\varkappa} + O^*\left(0.83226\frac{x^2}{\varkappa}\right) \geq 0.2263\frac{x^2}{\varkappa} \end{aligned}$$

for $r_0 = 150000$, where $x = N/(2+1/(36\sqrt{2\pi}))$, as in (8.11). (We are using (8.21) and (8.41).) Recall that $\varkappa = 49$ and $\eta_*(t) = (\eta_2 *_M \varphi)(\varkappa t)$, where $\varphi(t) = t^2 e^{-t^2/2}$.

It only remains to show that the contribution of terms with n_1, n_2 or n_3 non-prime to the sum in (8.42) is negligible. (Let us take out n_1, n_2, n_3 equal to 2 as well, since some prefer to state the ternary Goldbach conjecture as follows: every odd number ≥ 9 is the sum of three *odd* primes.) Clearly

$$\begin{aligned} &\sum_{\substack{n_1+n_2+n_3=N \\ n_1,\, n_2 \text{ or } n_3 \text{ even or non-prime}}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_+(n_1)\eta_+(n_2)\eta_*(n_3) \\ &\qquad \leq 3|\eta_+|_\infty^2 |\eta_*|_\infty \sum_{\substack{n_1+n_2+n_3=N \\ n_1 \text{ even or non-prime}}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) \\ &\qquad \leq 3|\eta_+|_\infty^2 |\eta_*|_\infty\cdot(\log N) \sum_{\substack{n_1 \leq N \text{ non-prime} \\ \text{or } n_1 = 2}} \Lambda(n_1) \sum_{n_2\leq N} \Lambda(n_2). \end{aligned} \tag{8.43}$$

By (8.1) and (8.15), $|\eta_+|_\infty \leq 1.0858$ and $|\eta_*|_\infty \leq 1.414$. By [RS62, Thms. 12 and 13],

$$\sum_{\substack{n_1 \leq N \text{ non-prime} \\ \text{or } n_1 = 2}} \Lambda(n_1) < 1.4262\sqrt{N},$$

$$\sum_{\substack{n_2 \leq N \text{ non-prime} \\ \text{or } n_2 = 2}} \Lambda(n_2) = 1.03883 \cdot 1.4262 N \leq 1.48158N.$$

Hence, the sum on the first line of (8.43) is at most

$$10.568 N^{3/2} \log N.$$

Thus, for $N \geq 10^{30}$ odd,

$$\begin{aligned}\sum_{\substack{n_1+n_2+n_3=N \\ n_1,\, n_2,\, n_3 \text{ odd primes}}} &\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_+(n_1)\eta_+(n_2)\eta_*(n_3) \\ &\geq 0.2263\frac{x^2}{\varkappa} - 1.0568 N^{3/2}\log N \\ &\geq 0.001141 N^2 - 8\cdot 10^{-14}\cdot N^2 \geq 0.00114 N^2\end{aligned}$$

by $\varkappa = 49$ and (8.11). Since $0.00114N^2 > 0$, this shows that every odd number $N \geq 10^{30}$ can be written as the sum of three odd primes.

Since the ternary Goldbach conjecture has already been checked for all $N \leq 10^{30}$ [HP], we conclude that every odd number $N > 7$ can be written as the sum of three odd primes, and every odd number $N > 5$ can be written as the sum of three primes. The main theorem is hereby proven: the ternary Goldbach conjecture is true.

Appendix A. Sums over primes

Here we treat some sums of the type $\sum_n \Lambda(n)\varphi(n)$, where φ has compact support. Since the sums are over all integers (not just an arithmetic progression) and there is no phase $e(\alpha n)$ involved, the treatment is relatively straightforward.

The following is standard.

Lemma A.1 (Explicit formula)**.** *Let $\varphi : [1,\infty) \to \mathbb{C}$ be continuous and piecewise C^1 with $\varphi'' \in L_1$; let it also be of compact support contained in $[1,\infty)$. Then*

$$\sum_n \Lambda(n)\varphi(n) = \int_1^\infty \left(1 - \frac{1}{x(x^2-1)}\right)\varphi(x)dx - \sum_\rho (M\varphi)(\rho), \tag{A.1}$$

where ρ runs over the non-trivial zeros of $\zeta(s)$.

The non-trivial zeros of $\zeta(s)$ are, of course, those in the critical strip $0 < \Re(s) < 1$.

Remark. Lemma A.1 appears as exercise 5 in [IK04, §5.5]; the condition there that φ be smooth can be relaxed, since already the weaker assumption that φ'' be in L^1 implies that the Mellin transform $(M\varphi)(\sigma+it)$ decays quadratically on t as $t\to\infty$, thereby guaranteeing that the sum $\sum_\rho (M\varphi)(\rho)$ converges absolutely.

Lemma A.2. *Let $x \geq 10$. Let η_2 be as in (4.33). Assume that all non-trivial zeros of $\zeta(s)$ with $|\Im(s)| \leq T_0$ lie on the critical line.*

Then

$$\sum_n \Lambda(n)\eta_2\left(\frac{n}{x}\right) = x + O^*\left(0.135x^{1/2} + \frac{9.7}{x^2}\right) + \frac{\log\frac{eT_0}{2\pi}}{T_0}\left(\frac{9/4}{2\pi} + \frac{6.03}{T_0}\right)x. \tag{A.2}$$

In particular, with $T_0 = 3.061\cdot 10^{10}$ in the assumption, we have, for $x \geq 2000$,

$$\sum_n \Lambda(n)\eta_2\left(\frac{n}{x}\right) = (1+O^*(\epsilon))x + O^*(0.135x^{1/2}),$$

where $\epsilon = 2.73\cdot 10^{-10}$.

The assumption that all non-trivial zeros up to $T_0 = 3.061\cdot 10^{10}$ lie on the critical line was proven rigorously in [Plaa]; higher values of T_0 have been reached elsewhere ([Wed03], [GD04]).

Proof. By Lemma A.1,

$$\sum_n \Lambda(n)\eta_2\left(\frac{n}{x}\right) = \int_1^\infty \eta_2\left(\frac{t}{x}\right)dt - \int_1^\infty \frac{\eta_2(t/x)}{t(t^2-1)}dt - \sum_\rho (M\varphi)(\rho),$$

where $\varphi(u) = \eta_2(u/x)$ and ρ runs over all non-trivial zeros of $\zeta(s)$. Since η_2 is non-negative, $\int_1^\infty \eta_2(t/x)dt = x|\eta_2|_1 = x$, while

$$\int_1^\infty \frac{\eta_2(t/x)}{t(t^2-1)}dt = O^*\left(\int_{1/4}^1 \frac{\eta_2(t)}{tx^2(t^2-1/100)}dt\right) = O^*\left(\frac{9.61114}{x^2}\right).$$

By (2.6),

$$\sum_\rho (M\varphi)(\rho) = \sum_\rho M\eta_2(\rho)\cdot x^\rho = \sum_\rho \left(\frac{1-2^{-\rho}}{\rho}\right)^2 x^\rho = S_1(x) - 2S_1(x/2) + S_1(x/4),$$

where

$$S_m(x) = \sum_\rho \frac{x^\rho}{\rho^{m+1}}. \tag{A.3}$$

Setting aside the contribution of all ρ with $|\Im(\rho)| \leq T_0$ and all ρ with $|\Im(\rho)| > T_0$ and $\Re(s) \leq 1/2$, and using the symmetry provided by the functional equation, we obtain

$$\begin{aligned}|S_m(x)| &\leq x^{1/2}\cdot\sum_\rho \frac{1}{|\rho|^{m+1}} + x\cdot \sum_{\substack{\rho\\ |\Im(\rho)|>T_0\\ |\Re(\rho)|>1/2}} \frac{1}{|\rho|^{m+1}}\\ &\leq x^{1/2}\cdot\sum_\rho \frac{1}{|\rho|^{m+1}} + \frac{x}{2}\cdot \sum_{\substack{\rho\\ |\Im(\rho)|>T_0}} \frac{1}{|\rho|^{m+1}}.\end{aligned}$$

We bound the first sum by [Ros41, Lemma 17] and the second sum by [RS03, Lemma 2]. We obtain

$$|S_m(x)| \leq \left(\frac{1}{2m\pi T_0^m} + \frac{2.68}{T_0^{m+1}}\right) x\log\frac{eT_0}{2\pi} + \kappa_m x^{1/2}, \tag{A.4}$$

where $\kappa_1 = 0.0463$, $\kappa_2 = 0.00167$ and $\kappa_3 = 0.0000744$.

Hence

$$\left|\sum_\rho (M\eta)(\rho)\cdot x^\rho\right| \leq \left(\frac{1}{2\pi T_0} + \frac{2.68}{T_0^2}\right)\frac{9x}{4}\log\frac{eT_0}{2\pi} + \left(\frac{3}{2}+\sqrt{2}\right)\kappa_1 x^{1/2}.$$

For $T_0 = 3.061 \cdot 10^{10}$ and $x \geq 2000$, we obtain

$$\sum_n \Lambda(n)\eta_2\left(\frac{n}{x}\right) = (1 + O^*(\epsilon))x + O^*(0.135x^{1/2}),$$

where $\epsilon = 2.73 \cdot 10^{-10}$. □

Corollary A.3. *Let η_2 be as in (4.33). Assume that all non-trivial zeros of $\zeta(s)$ with $|\Im(s)| \leq T_0$, $T_0 = 3.061 \cdot 10^{10}$, lie on the critical line. Then, for all $x \geq 1$,*

$$\sum_n \Lambda(n)\eta_2\left(\frac{n}{x}\right) \leq \min\left((1+\epsilon)x + 0.2x^{1/2}, 1.04488x\right), \tag{A.5}$$

where $\epsilon = 2.73 \cdot 10^{-10}$.

Proof. Immediate from Lemma A.2 for $x \geq 2000$. For $x < 2000$, we use computation as follows. Since $|\eta_2'|_\infty = 16$ and $\sum_{x/4 \leq n \leq x} \Lambda(n) \leq x$ for all $x \geq 0$, computing $\sum_{n \leq x} \Lambda(n)\eta_2(n/x)$ only for $x \in (1/1000)\mathbb{Z} \cap [0, 2000]$ results in an inaccuracy of at most $(16 \cdot 0.0005/0.9995)x \leq 0.00801x$. This resolves the matter at all points outside $(205, 207)$ (for the first estimate) or outside $(9.5, 10.5)$ and $(13.5, 14.5)$ (for the second estimate). In those intervals, the prime powers n involved do not change (since whether $x/4 < n \leq x$ depends only on n and $[x]$), and thus we can find the maximum of the sum in (A.5) just by taking derivatives. □

Appendix B. Sums involving $\phi(q)$

We need estimates for several sums involving $\phi(q)$ in the denominator.

The easiest are convergent sums, such as $\sum_q \mu^2(q)/(\phi(q)q)$. We can express this as $\prod_p (1 + 1/(p(p-1)))$. This is a convergent product, and the main task is to bound a tail: for r an integer,

$$\log \prod_{p>r}\left(1 + \frac{1}{p(p-1)}\right) \leq \sum_{p>r} \frac{1}{p(p-1)} \leq \sum_{n>r} \frac{1}{n(n-1)} = \frac{1}{r}. \tag{B.1}$$

A quick computation[12] now suffices to give

$$2.591461 \leq \sum_q \frac{\gcd(q,2)\mu^2(q)}{\phi(q)q} < 2.591463 \tag{B.2}$$

and so

$$1.295730 \leq \sum_{q \text{ odd}} \frac{\mu^2(q)}{\phi(q)q} < 1.295732, \tag{B.3}$$

since the expression bounded in (B.3) is exactly half of that bounded in (B.2).

Again using (B.1), we get that

$$2.826419 \leq \sum_q \frac{\mu^2(q)}{\phi(q)^2} < 2.826421. \tag{B.4}$$

In what follows, we will use values for convergent sums obtained in much the same way – an easy tail bound followed by a computation.

[12]Using D. Platt's integer arithmetic package.

By [Ram95, Lemma 3.4],

$$\sum_{q\leq r} \frac{\mu^2(q)}{\phi(q)} = \log r + c_E + O^*(7.284 r^{-1/3}),$$

$$\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} = \frac{1}{2}\left(\log r + c_E + \frac{\log 2}{2}\right) + O^*(4.899 r^{-1/3}), \tag{B.5}$$

where

$$c_E = \gamma + \sum_p \frac{\log p}{p(p-1)} = 1.332582275 + O^*(10^{-9}/3)$$

by [RS62, (2.11)]. As we already said in (7.15), this, supplemented by a computation for $r \leq 4\cdot 10^7$, gives

$$\log r + 1.312 \leq \sum_{q\leq r} \frac{\mu^2(q)}{\phi(q)} \leq \log r + 1.354$$

for $r \geq 182$. In the same way, we get that

$$\frac{1}{2}\log r + 0.83 \leq \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)} \leq \frac{1}{2}\log r + 0.85. \tag{B.6}$$

for $r \geq 195$. (The numerical verification here goes up to $1.38\cdot 10^8$; for $r > 3.18\cdot 10^8$, use B.6.)

Clearly

$$\sum_{\substack{q\leq 2r\\ q \text{ even}}} \frac{\mu^2(q)}{\phi(q)} = \sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)}. \tag{B.7}$$

We wish to obtain bounds for the sums

$$\sum_{q\geq r} \frac{\mu^2(q)}{\phi(q)^2}, \quad \sum_{\substack{q\geq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)^2}, \quad \sum_{\substack{q\geq r\\ q \text{ even}}} \frac{\mu^2(q)}{\phi(q)^2},$$

where $N \in \mathbb{Z}^+$ and $r \geq 1$. To do this, it will be helpful to express some of the quantities within these sums as convolutions.[13] For q squarefree and $j \geq 1$,

$$\frac{\mu^2(q) q^{j-1}}{\phi(q)^j} = \sum_{ab=q} \frac{f_j(b)}{a}, \tag{B.8}$$

where f_j is the multiplicative function defined by

$$f_j(p) = \frac{p^j - (p-1)^j}{(p-1)^j p}, \qquad f_j(p^k) = 0 \quad \text{for } k \geq 2.$$

We will also find the following estimate useful.

Lemma B.1. *Let $j \geq 2$ be an integer and A a positive real. Let $m \geq 1$ be an integer. Then*

$$\sum_{\substack{a\geq A\\ (a,m)=1}} \frac{\mu^2(a)}{a^j} \leq \frac{\zeta(j)/\zeta(2j)}{A^{j-1}} \cdot \prod_{p|m} \left(1 + \frac{1}{p^j}\right)^{-1}. \tag{B.9}$$

[13] The author would like to thank O. Ramaré for teaching him this technique.

It is useful to note that $\zeta(2)/\zeta(4) = 15/\pi^2 = 1.519817\ldots$ and $\zeta(3)/\zeta(6) = 1.181564\ldots$.

Proof. The right side of (B.9) decreases as A increases, while the left side depends only on $\lceil A \rceil$. Hence, it is enough to prove (B.9) when A is an integer.

For $A = 1$, (B.9) is an equality. Let

$$C = \frac{\zeta(j)}{\zeta(2j)} \cdot \prod_{p|m} \left(1 + \frac{1}{p^j}\right)^{-1}.$$

Let $A \geq 2$. Since

$$\sum_{\substack{a \geq A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} = C - \sum_{\substack{a < A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j}$$

and

$$\begin{aligned} C = \sum_{\substack{a \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} &< \sum_{\substack{a < A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} + \frac{1}{A^j} + \int_A^\infty \frac{1}{t^j} dt \\ &= \sum_{\substack{a < A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} + \frac{1}{A^j} + \frac{1}{(j-1)A^{j-1}}, \end{aligned}$$

we obtain

$$\begin{aligned} \sum_{\substack{a \geq A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} &= \frac{1}{A^{j-1}} \cdot C + \frac{A^{j-1}-1}{A^{j-1}} \cdot C - \sum_{\substack{a < A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} \\ &< \frac{C}{A^{j-1}} + \frac{A^{j-1}-1}{A^{j-1}} \cdot \left(\frac{1}{A^j} + \frac{1}{(j-1)A^{j-1}}\right) - \frac{1}{A^{j-1}} \sum_{\substack{a < A \\ (a,m)=1}} \frac{\mu^2(a)}{a^j} \\ &\leq \frac{C}{A^{j-1}} + \frac{1}{A^{j-1}} \left(\left(1 - \frac{1}{A^{j-1}}\right)\left(\frac{1}{A} + \frac{1}{j-1}\right) - 1\right). \end{aligned}$$

Since $(1 - 1/A)(1/A + 1) < 1$ and $1/A + 1/(j-1) \leq 1$ for $j \geq 3$, we obtain that

$$\left(1 - \frac{1}{A^{j-1}}\right)\left(\frac{1}{A} + \frac{1}{j-1}\right) < 1$$

for all integers $j \geq 2$, and so the statement follows. □

We now obtain easily the estimates we want: by (B.8) and Lemma B.1 (with $j = 2$ and $m = 1$),

$$\begin{aligned} \sum_{q \geq r} \frac{\mu^2(q)}{\phi(q)^2} &= \sum_{q \geq r} \sum_{ab=q} \frac{f_2(b)}{a} \frac{\mu^2(q)}{q} \leq \sum_{b \geq 1} \frac{f_2(b)}{b} \sum_{a \geq r/b} \frac{\mu^2(a)}{a^2} \\ &\leq \frac{\zeta(2)/\zeta(4)}{r} \sum_{b \geq 1} f_2(b) = \frac{\frac{15}{\pi^2}}{r} \prod_p \left(1 + \frac{2p-1}{(p-1)^2 p}\right) \leq \frac{6.7345}{r}. \end{aligned} \tag{B.10}$$

Similarly, by (B.8) and Lemma B.1 (with $j = 2$ and $m = 2$),

$$\begin{aligned}\sum_{\substack{q\geq r\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)^2} &= \sum_{\substack{b\geq 1\\ b \text{ odd}}} \frac{f_2(b)}{b} \sum_{\substack{a\geq r/b\\ a \text{ odd}}} \frac{\mu^2(a)}{a^2} \leq \frac{\zeta(2)/\zeta(4)}{1+1/2^2}\frac{1}{r}\sum_{b \text{ odd}} f_2(b)\\ &= \frac{12}{\pi^2}\frac{1}{r}\prod_{p>2}\left(1+\frac{2p-1}{(p-1)^2p}\right) \leq \frac{2.15502}{r}\end{aligned} \tag{B.11}$$

$$\sum_{\substack{q\geq r\\ q \text{ even}}} \frac{\mu^2(q)}{\phi(q)^2} = \sum_{\substack{q\geq r/2\\ q \text{ odd}}} \frac{\mu^2(q)}{\phi(q)^2} \leq \frac{4.31004}{r}. \tag{B.12}$$

Lastly,

(B.13)

$$\begin{aligned}\sum_{\substack{q\leq r\\ q \text{ odd}}} \frac{\mu^2(q)q}{\phi(q)} &= \sum_{\substack{q\leq r\\ q \text{ odd}}} \mu^2(q) \sum_{d|q} \frac{1}{\phi(d)} = \sum_{\substack{d\leq r\\ d \text{ odd}}} \frac{1}{\phi(d)} \sum_{\substack{q\leq r\\ d|q\\ q \text{ odd}}} \mu^2(q) \leq \sum_{\substack{d\leq r\\ d \text{ odd}}} \frac{1}{2\phi(d)}\left(\frac{r}{d}+1\right)\\ &\leq \frac{r}{2}\sum_{d \text{ odd}} \frac{1}{\phi(d)d} + \frac{1}{2}\sum_{\substack{d\leq r\\ d \text{ odd}}} \frac{1}{\phi(d)} \leq 0.64787r + \frac{\log r}{4} + 0.425,\end{aligned}$$

where we are using (B.3) and (B.6).

Appendix C. Validated numerics

C.1. Integrals of a smoothing function.

Let

$$h : t \mapsto \begin{cases} x^3(2-x)^3e^{x-1/2} & \text{if } t \in [0,2],\\ 0 & \text{otherwise}\end{cases} \tag{C.1}$$

Clearly, $h(0) = h'(0) = h''(0) = h(2) = h'(2) = h''(2) = 0$, and $h(x)$, $h'(x)$ and $h''(x)$ are all continuous. We are interested in computing

$$C_k = \int_0^\infty |h^{(k)}(x)|x^{k-1}dx$$

for $0 \leq k \leq 4$. (If $k = 4$, the integral is to be understood in the sense of distributions.)

Rigorous numerical integration[14] gives that

$$1.6222831573801406 \leq C_0 \leq 1.6222831573801515. \tag{C.2}$$

We will compute C_k, $1 \leq k \leq 4$, in a somewhat different way.[15]

The function $(x^3(2-x)^3e^{x-1/2})' = ((x^3(2-x)^3)' + x^3(2-x)^3)e^{x-1/2}$ has the same zeros as $H_1(x) = (x^3(2-x)^3)' + x^3(2-x)^3$, namely, 0, 2, $\sqrt{10}-2$ and $-(\sqrt{10}+2)$; only the first three of these four roots lie in the interval $[0,2]$.

[14] By VNODE-LP [Ned06], running on PROFIL/BIAS [Knü99].

[15] It does not seem possible to use [Ned06] directly on an integrand involving the abs function, due to the fact that it is not differentiable at the origin.

The sign of $H_1(x)$ (and hence of $h'(x)$) is $+$ within $(0, \sqrt{10}-2)$ and $-$ within $(\sqrt{10}-2, 2)$. Hence

$$C_1 = \int_0^\infty |h'(x)| dx = |h(\sqrt{10}-2) - h(0)| + |h(2) - h(\sqrt{10}-2)| \\ = 2h(\sqrt{10}-2) = 3.58000383169 + O^*\left(10^{-12}\right). \tag{C.3}$$

The situation with $(x^3(2-x)^3 e^{x-1/2})''$ is similar: it has zeros at the roots of $H_2(x) = 0$, where $H_2(x) = H_1(x) + H_1'(x)$ (and, in general, $H_{k+1}(x) = H_k(x) + H_k'(x)$). The roots within $[0, 2]$ are 0, $\sqrt{3}-1$, $\sqrt{21}-3$ and 2. Write $\alpha_{2,1} = \sqrt{3}-1$, $\alpha_{2,2} = \sqrt{21}-3$. The sign of $H_2(x)$ (and hence of $h''(x)$) is first $+$, then $-$, then $+$. Hence

(C.4)

$$\begin{aligned} C_2 = \int_0^\infty |h''(x)| x dx &= \int_0^{\alpha_{2,1}} h''(x) x dx - \int_{\alpha_{2,1}}^{\alpha_{2,2}} h''(x) x dx + \int_{\alpha_{2,2}}^2 h''(x) x dx \\ &= h'(x) x|_0^{\alpha_{2,1}} - \int_0^{\alpha_{2,1}} h'(x) dx - \left(h'(x) x|_{\alpha_{2,1}}^{\alpha_{2,2}} - \int_{\alpha_{2,1}}^{\alpha_{2,2}} h'(x) dx \right) \\ &+ h'(x) x|_{\alpha_{2,2}}^2 - \int_{\alpha_{2,2}}^2 h'(x) dx \\ &= 2h'(\alpha_{2,1})\alpha_{2,1} - 2h(\alpha_{2,1}) - 2h'(\alpha_{2,2})\alpha_{2,2} + 2h(\alpha_{2,2}) \\ &= 15.27956091266 + O^*\left(10^{-11}\right). \end{aligned}$$

To compute C_3, we proceed in the same way, except now we must find the roots numerically. It is enough to find (candidates for) the roots using any available tool[16] and then check rigorously that the sign does change around the purported roots. In this way, we check that the roots $\alpha_{3,1}$, $\alpha_{3,2}$, $\alpha_{3,3}$ of $H_3(x) = 0$ lie within the intervals

$$[0.366931547524632, 0.366931547524633],$$
$$[1.233580882085861, 1.233580882085862],$$
$$[1.847147885393624, 1.847147885393625],$$

respectively. The sign of $H_3(x)$ on the interval $[0, 2]$ is first $+$, then $-$, then $+$, then $-$. Proceeding as before, we obtain that

$$C_3 = \int_0^\infty |h'''(x)| x^2 dx \\ = 2 \sum_{j=1}^3 (-1)^{j+1} (h''(\alpha_{3,j})\alpha_{3,j}^2 - 2h'(\alpha_{3,j})\alpha_{3,j} + 2h(\alpha_{3,j})) \tag{C.5}$$

and so interval arithmetic gives us

$$C_3 = 131.3398196149 + O^*\left(10^{-10}\right). \tag{C.6}$$

The treatment of the integral in C_4 is very similar, at least as first. The roots $\alpha_{4,1}$, $\alpha_{4,2}$ of $H_4(x) = 0$ lie within the intervals

$$[0.866114027542349, 0.86611402754235],$$
$$[1.640243631518005, 1.640243631518006].$$

[16] Routine `find_root` in SAGE was used here.

The sign of $H_4(x)$ on the interval $[0,2]$ is first $-$, then $+$, then $-$. Using integration by parts as before, we obtain

$$\int_0^{2^-} \left|h^{(4)}(x)\right| x^3 dx = \lim_{x\to 0^+} h^{(3)}(x)x^3 - \lim_{x\to 2^-} h^{(3)}(x)x^3$$
$$+ 2\sum_{j=1}^{2}(-1)^j \left(h^{(3)}(\alpha_{4,j})\alpha_{4,j}^3 - 3h''(\alpha_{4,j})\alpha_{4,j}^2 + 6h'(\alpha_{4,j})\alpha_{4,j} - 6h(\alpha_{4,j})\right)$$
$$= 2199.91310061863 + O\left(3\cdot 10^{-11}\right).$$

Now

$$\int_{2^-}^{\infty} |h^{(4)}(x)x^3|dx = \lim_{\epsilon\to 0^+} |h^{(3)}(2+\epsilon) - h^{(3)}(2-\epsilon)|\cdot 2^3 = 48\cdot e^{3/2}\cdot 2^3.$$

Hence

$$C_{4,\sigma} = 48\cdot e^{3/2}\cdot 2^3 = 3920.8817036284 + O\left(10^{-10}\right). \tag{C.7}$$

C.2. **Extrema via bisection and truncated series.** Let $f: I\to\mathbb{R}$, $I\subset\mathbb{R}$. We wish to find the minima and maxima of f in I rigorously.

The bisection method (as described in, e.g., [Tuc11, §5.2]) can be used to show that the minimum (or maximum) of f on a compact interval I lies within an interval (usually a very small one). We will need to complement it by other arguments if either (a) I is not compact, or (b) we want to know the minimum or maximum exactly.

As in §5.3, let $j(\rho) = (1+\rho^2)^{1/2}$ and $\upsilon(\rho) = \sqrt{(1+j(\rho))/2}$ for $\rho\geq 0$. Let Υ, $\cos\theta_0$, $\sin\theta_0$, c_0 and c_1 be understood as one-variable real-valued functions on ρ, given by (3.13), (5.25) and (5.31).

First, let us bound $\Upsilon(\rho)$ from below. By the bisection method[17] applied with 32 iterations,

$$0.798375987 \leq \min_{0\leq\rho\leq 10} \Upsilon(\rho) \leq 0.798375989.$$

Since $j(\rho)\geq\rho$ and $\upsilon(\rho)\geq\sqrt{j(\rho)/2}\geq\sqrt{\rho/2}$,

$$0 \leq \frac{\rho}{2\upsilon(\rho)(\upsilon(\rho)+j(\rho))} \leq \frac{\rho}{\sqrt{2}\rho^{3/2}} = \frac{1}{\sqrt{2\rho}},$$

and so

$$\Upsilon(\rho) \geq 1 - \frac{\rho}{2\upsilon(\rho)(\upsilon(\rho)+j(\rho))} \geq 1 - \frac{1}{\sqrt{2\rho}}. \tag{C.8}$$

Hence $\Upsilon(\rho)\geq 0.8418$ for $\rho\geq 20$. We conclude that

$$0.798375987 \leq \min_{\rho\geq 0} \Upsilon(\rho) \leq 0.798375989. \tag{C.9}$$

Now let us bound $c_0(\rho)$ from below. For $\rho\geq 8$,

$$\sin\theta_0 = \sqrt{\frac{1}{2} - \frac{1}{2\upsilon}} \geq \sqrt{\frac{1}{2} - \frac{1}{\sqrt{2\rho}}} \geq \frac{1}{2},$$

whereas $\cos\theta_0\geq 1/\sqrt{2}$ for all $\rho\geq 0$. Hence, by (C.9)

$$c_0(\rho) \geq \frac{0.7983}{\sqrt{2}} + \frac{1}{2} > 1.06 \tag{C.10}$$

[17]Implemented by the author from the description in [Tuc11, p. 87–88], using D. Platt's interval arithmetic package.

for $\rho \geq 8$. The bisection method applied with 28 iterations gives us that

$$\max_{0.01 \leq \rho \leq 8} c_0(\rho) \geq 1 + 5 \cdot 10^{-8} > 1. \tag{C.11}$$

It remains to study $c_0(\rho)$ for $\rho \in [0, 0.01]$. The method we are about to give actually works for all $\rho \in [0,1]$.

Since

$$(\sqrt{1+x})' = \frac{1}{2\sqrt{1+x}}, \qquad (\sqrt{1+x})'' = -\frac{1}{4(1+x)^{3/2}},$$
$$\left(\frac{1}{\sqrt{1+x}}\right)' = \frac{-1}{2(1+x)^{3/2}}, \qquad \left(\frac{1}{\sqrt{1+x}}\right)'' = \left(\frac{-1/2}{(1+x)^{3/2}}\right)' = \frac{3/4}{(1+x)^{5/2}},$$

a truncated Taylor expansion gives us that, for $x \geq 0$,

$$\begin{aligned} 1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x \\ 1 - \frac{1}{2}x \leq \frac{1}{\sqrt{1+x}} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2. \end{aligned} \tag{C.12}$$

Hence, for $\rho \geq 0$,

$$\begin{aligned} 1 + \rho^2/2 - \rho^4/8 \leq j(\rho) \leq 1 + \rho^2/2, \\ 1 + \rho^2/8 - 5\rho^4/128 + \rho^6/256 - \rho^8/2048 \leq \upsilon(\rho) \leq 1 + \rho^2/8, \end{aligned} \tag{C.13}$$

and so

$$\upsilon(\rho) \geq 1 + \rho^2/8 - 5\rho^4/128 \tag{C.14}$$

for $\rho \leq 8$. We also get from (C.12) that

$$\begin{aligned} \frac{1}{\upsilon(\rho)} &= \frac{1}{\sqrt{1 + \frac{j(\rho)-1}{2}}} \leq 1 - \frac{1}{2}\frac{j(\rho)-1}{2} + \frac{3}{8}\left(\frac{j(\rho)-1}{2}\right)^2 \\ &\leq 1 - \frac{1}{2}\left(\frac{\rho^2}{4} - \frac{\rho^4}{16}\right) + \frac{3}{8}\frac{\rho^4}{16} \leq 1 - \frac{\rho^2}{8} + \frac{7\rho^4}{128}, \\ \frac{1}{\upsilon(\rho)} &= \frac{1}{\sqrt{1 + \frac{j(\rho)-1}{2}}} \geq 1 - \frac{1}{2}\frac{j(\rho)-1}{2} \geq 1 - \frac{\rho^2}{8}. \end{aligned} \tag{C.15}$$

Hence

$$\begin{aligned} \sin\theta_0 &= \sqrt{\frac{1}{2} - \frac{1}{2\upsilon(\rho)}} \geq \sqrt{\frac{\rho^2}{16} - \frac{7\rho^4}{256}} = \frac{\rho}{4}\sqrt{1 - \frac{7}{16}\rho^2}, \\ \sin\theta_0 &\leq \sqrt{\frac{\rho^2}{16}} = \frac{\rho}{4}, \end{aligned} \tag{C.16}$$

while

$$\cos\theta_0 = \sqrt{\frac{1}{2} + \frac{1}{2\upsilon(\rho)}} \geq \sqrt{1 - \frac{\rho^2}{16}}, \qquad \cos\theta_0 \leq \sqrt{1 - \frac{\rho^2}{16} + \frac{7\rho^4}{256}}, \tag{C.17}$$

By (C.13) and (C.15),

$$\frac{\rho}{2\upsilon(\upsilon + j)} \geq \frac{\rho}{2}\frac{1 - \rho^2/8}{2 + 5\rho^2/8} \geq \frac{\rho}{2}\left(\frac{1}{2} - \frac{3\rho^2}{32}\right) = \frac{\rho}{4} - \frac{3\rho^3}{64}. \tag{C.18}$$

Assuming $0 \leq \rho \leq 1$,

$$\frac{1}{1+\frac{5\rho^2}{16}-\frac{9\rho^4}{64}} \leq \left(1-\frac{5\rho^2}{16}+\frac{9\rho^4}{64}+\left(\frac{5\rho^2}{16}-\frac{9\rho^4}{64}\right)^2\right) \leq 1-\frac{5\rho^2}{16}+\frac{61\rho^4}{256},$$

and so, by (C.14) and (C.15),

$$\begin{aligned}\frac{\rho}{2\upsilon(\upsilon+j)} &\leq \frac{\rho}{2}\frac{1-\frac{\rho^2}{8}+\frac{7\rho^4}{128}}{2+\frac{5\rho^2}{8}-\frac{21\rho^4}{128}}\\ &\leq \frac{\rho}{4}\left(1-\frac{\rho^2}{8}+\frac{7\rho^4}{128}\right)\left(1-\frac{5\rho^2}{16}+\frac{46\rho^4}{256}\right)\\ &\leq \frac{\rho}{4}\left(1-\frac{7\rho^2}{16}+\frac{35}{128}\rho^4-\frac{81}{2048}\rho^6+\frac{161}{2^{14}}\rho^8\right) \leq \frac{\rho}{4}-\frac{7\rho^3}{64}+\frac{35\rho^5}{512}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Upsilon(\rho) &= \sqrt{1+\left(\frac{\rho}{2\upsilon(\upsilon+j)}\right)^2}-\frac{\rho}{2\upsilon(\upsilon+j)}\\ &\geq 1+\frac{1}{2}\left(\frac{\rho}{4}-\frac{3\rho^3}{64}\right)^2-\frac{1}{8}\left(\frac{\rho^2}{16}\right)^2-\left(\frac{\rho}{4}-\frac{7\rho^3}{64}+\frac{35\rho^5}{512}\right)\\ &\geq 1-\frac{\rho}{4}+\frac{\rho^2}{32}+\frac{7\rho^3}{64}-\left(\frac{3}{256}+\frac{1}{2048}\right)\rho^4-\frac{35\rho^5}{512}+\frac{9\rho^6}{2^{13}}\\ &\geq 1-\frac{\rho}{4}+\frac{\rho^2}{32}+\frac{7\rho^3}{64}-\frac{165\rho^4}{2048},\end{aligned} \tag{C.19}$$

where, in the last line, we use again the assumption $\rho \leq 1$.

For $x \in [-1/4, 0]$,

$$\begin{aligned}\sqrt{1+x} &\geq 1+\frac{1}{2}x-\frac{x^2}{2}\frac{1}{4(1-1/4)^{3/2}} = 1+\frac{x}{2}-\frac{x^2}{3^{3/2}}\\ \sqrt{1+x} &\leq 1+\frac{1}{2}x-\frac{x^2}{8} \leq 1+\frac{1}{2}x.\end{aligned}$$

Hence

$$\begin{aligned}1-\frac{\rho^2}{32}-\frac{\rho^4}{3^{3/2}\cdot 256} \leq \cos\theta_0 &\leq \sqrt{1-\frac{\rho^2}{16}+\frac{7\rho^4}{256}} \leq 1-\frac{\rho^2}{32}+\frac{7\rho^4}{512}\\ \frac{\rho}{4}\left(1-\frac{7}{32}\rho^2-\frac{49}{3^{3/2}\cdot 256}\rho^4\right) \leq \sin\theta_0 &\leq \frac{\rho}{4}\end{aligned} \tag{C.20}$$

for $\rho \leq 1$. Therefore,

$$\begin{aligned}c_0(\rho) &= \Upsilon(\rho)\cdot\cos\theta_0+\sin\theta_0\\ &\geq \left(1-\frac{\rho}{4}+\frac{\rho^2}{32}+\frac{7\rho^3}{64}-\frac{165}{2048}\rho^4\right)\left(1-\frac{\rho^2}{32}-\frac{\rho^4}{3^{3/2}\cdot 256}\right)\\ &+\frac{\rho}{4}-\frac{7}{128}\rho^3-\frac{49}{3^{3/2}\cdot 1024}\rho^5\\ &\geq 1+\frac{\rho^3}{16}-\left(\left(\frac{\sqrt{3}}{2304}+\frac{167}{2048}\right)+\left(\frac{7}{2048}+\frac{\sqrt{3}}{192}\right)\rho+\frac{7\sqrt{3}}{147456}\rho^3\right)\rho^4\\ &\geq 1+\frac{\rho^3}{16}-0.0949\rho^4,\end{aligned}$$

where we are again using $\rho \leq 1$. We conclude that, for all $\rho \in (0, 1/2]$,

$$c_0(\rho) > 1.$$

Together with (C.10) and (C.11), this shows that

$$c_0(\rho) > 1 \qquad \forall \rho > 0. \tag{C.21}$$

It is easy to check that $c_0(0) = 1$.

(The truncated-Taylor series estimates above could themselves have been done automatically; see [Tuc11, Ch. 4] (automatic differentiation). The footnote in [Tuc11, p. 72] (referring to the work of Berz and Makino [BM98] on "Taylor models") seems particularly relevant here. We have preferred to do matters "by hand" in the above.)

Now let us examine $\eta(\rho)$, given as in (5.43). Let us first focus on the case of ρ large. We can use the lower bound (C.8) on $\Upsilon(\rho)$. To obtain a good upper bound on $\Upsilon(\rho)$, we need to get truncated series expansions on $1/\rho$ for υ and j. These are:

(C.22)
$$\begin{aligned} j(\rho) &= \sqrt{\rho^2+1} = \rho\sqrt{1+\frac{1}{\rho^2}} \leq \rho\left(1+\frac{1}{2\rho^2}\right) = \rho + \frac{1}{2\rho}, \\ \upsilon(\rho) &= \sqrt{\frac{1+j}{2}} \leq \sqrt{\frac{\rho}{2}+\frac{1}{2}+\frac{1}{4\rho}} = \sqrt{\frac{\rho}{2}}\sqrt{1+\frac{1}{\rho}+\frac{1}{2\rho^2}} \leq \sqrt{\frac{\rho}{2}}\left(1+\frac{1}{\sqrt{2}\rho}\right), \end{aligned}$$

together with the trivial bounds $j(\rho) \geq \rho$ and $\upsilon(\rho) \geq \sqrt{j(\rho)/2} \geq \sqrt{\rho/2}$. By (C.22),

$$\begin{aligned} \frac{1}{\upsilon^2-\upsilon} &\geq \frac{1}{\frac{\rho}{2}\left(1+\frac{1}{\sqrt{2}\rho}\right)^2 - \sqrt{\frac{\rho}{2}}} = \frac{\left(1+\sqrt{\frac{2}{\rho}}\right)}{\frac{\rho}{2}\left(\left(1+\frac{1}{\sqrt{2}\rho}\right)^2 - \sqrt{\frac{2}{\rho}}\right)\left(1+\sqrt{\frac{2}{\rho}}\right)} \\ &= \frac{\frac{2}{\rho}\left(1+\sqrt{\frac{2}{\rho}}\right)}{1-\frac{2}{\rho}+\left(\frac{\sqrt{2}}{\rho}+\frac{1}{2\rho^2}\right)\left(1+\sqrt{\frac{2}{\rho}}\right)} \geq \frac{2}{\rho}+\frac{\sqrt{8}}{\rho^{3/2}} \end{aligned} \tag{C.23}$$

for $\rho \geq 15$, and so

$$\frac{j}{\upsilon^2-\upsilon} \geq 2 + \sqrt{\frac{8}{\rho}} \tag{C.24}$$

for $\rho \geq 15$. In fact, the bisection method (applied with 20 iterations, including 10 "initial" iterations after which the possibility of finding a minimum within each interval is tested) shows that (C.23) (and hence (C.24)) holds for all $\rho \geq 1$. By (C.22),

$$\begin{aligned} \frac{\rho}{2\upsilon(\upsilon+j)} &\geq \frac{\rho}{\sqrt{2\rho}\left(1+\frac{1}{\sqrt{2}\rho}\right)\left(\sqrt{\frac{\rho}{2}}\left(1+\frac{1}{\sqrt{2}\rho}\right)+\rho+\frac{1}{2\rho}\right)} \\ &\geq \frac{1}{\sqrt{2\rho}}\cdot\frac{1}{1+\frac{1}{\sqrt{2\rho}}+\frac{1}{\rho}} \geq \frac{1}{\sqrt{2\rho}}-\frac{1}{2\rho}-\frac{1}{\sqrt{2}\rho^{3/2}} \end{aligned} \tag{C.25}$$

for $\rho \geq 16$. (Again, (C.25) is also true for $1 \leq \rho \leq 16$ by the bisection method; it is trivially true for $\rho \in [0, 1]$, since the last term of (C.25) is then negative.) We also have the easy upper bound

$$\frac{\rho}{2\upsilon(\upsilon+j)} \leq \frac{\rho}{2\cdot\sqrt{\frac{\rho}{2}}\cdot(\sqrt{\frac{\rho}{2}}+\rho)} = \frac{1}{\sqrt{2\rho}+1} \leq \frac{1}{\sqrt{2\rho}} - \frac{1}{2\rho} + \frac{1}{(2\rho)^{3/2}} \tag{C.26}$$

valid for $\rho \geq 1/2$.

Hence, by (C.12), (C.25) and (C.26),

$$\begin{aligned}\Upsilon &= \sqrt{1+\left(\frac{\rho}{2\upsilon(\upsilon+j)}\right)^2} - \frac{\rho}{2\upsilon(\upsilon+j)}\\ &\leq 1 + \frac{1}{2}\left(\frac{1}{\sqrt{2\rho}}-\frac{1}{2\rho}\right)^2 - \frac{1}{\sqrt{2\rho}} + \frac{1}{2\rho} + \frac{1}{\sqrt{2}\rho^{3/2}} \leq 1 - \frac{1}{\sqrt{2\rho}} + \frac{1}{\rho}\end{aligned}$$

for $\rho \geq 3$. Again, we use the bisection method (with 20 iterations) on $[1/2, 3]$, and note that $1/\sqrt{2\rho} < 1/\rho$ for $\rho < 1/2$; we thus obtain

$$\Upsilon \leq 1 - \frac{1}{\sqrt{2\rho}} + \frac{1}{\rho} \tag{C.27}$$

for all $\rho > 0$.

We recall (5.43) and the lower bounds (C.24) and (C.8). We get

$$\begin{aligned}\eta &\geq \frac{1}{\sqrt{2}}\sqrt{2+\sqrt{\frac{8}{\rho}}}\left(1+\left(1-\frac{1}{\sqrt{2\rho}}\right)^2\right) - \frac{1}{2}\left(1-\frac{1}{\sqrt{2\rho}}+\frac{1}{\rho}\right)^2\\ &\quad + \frac{1}{2} - \frac{1}{\sqrt{2\rho}} - \frac{\rho}{\rho+1}\cdot\left(1-\frac{1}{\sqrt{2\rho}}+\frac{1}{\rho}\right)\\ &\geq \left(1+\frac{1}{\sqrt{2\rho}}-\frac{1}{4\rho}\right)\left(2-\frac{\sqrt{2}}{\sqrt{\rho}}+\frac{1}{2\rho}\right) - \frac{1}{2}\left(1-\frac{2}{\sqrt{2\rho}}+\frac{5}{2\rho}\right)\\ &\quad + \frac{1}{2} - \frac{1}{\sqrt{2\rho}} - \left(1-\rho^{-1}+\rho^{-2}\right)\left(1-\frac{1}{\sqrt{2\rho}}+\frac{1}{\rho}\right)\\ &\geq 1 + \frac{1}{\sqrt{2\rho}} - \frac{9}{4\rho} - \frac{1}{8\rho^2} + \frac{1}{\sqrt{2}\rho^{5/2}} - \frac{1}{\rho^3} \geq 1 + \frac{1}{\sqrt{2\rho}} - \frac{37}{16\rho}\end{aligned} \tag{C.28}$$

for $\rho \geq 2$. This implies that $\eta(\rho) > 1$ for $\rho \geq 11$. (Since our estimates always give an error of at most $O(1/\sqrt{\rho})$, we also get $\lim_{\rho\to\infty}\eta(\rho) = 1$.) The bisection method (with 20 iterations, including 6 initial iterations) gives that $\eta(\rho) > 1$ also holds for $1 \leq \rho \leq 11$.

Let us now look at what happens for $\rho \leq 1$. From (C.19), we get the simpler bound

$$\Upsilon \geq 1 - \frac{\rho}{4} + \frac{\rho^2}{32} + \frac{3\rho^3}{32} \geq 1 - \frac{\rho}{4} \tag{C.29}$$

valid for $\rho \leq 1$, implying that

$$\Upsilon^2 \geq 1 - \frac{\rho}{2} + \frac{\rho^2}{8} + \frac{11\rho^3}{64} - \frac{23\rho^4}{1024}$$

for $\rho \leq 1$. We also have, by (5.23) and (C.18),

$$\begin{aligned}\Upsilon &\leq 1 + \frac{1}{2}\left(\frac{\rho}{2\upsilon(\upsilon+j)}\right)^2 - \frac{\rho}{2\upsilon(\upsilon+j)} \leq 1 + \frac{1}{2}\left(\frac{\rho}{4}\right)^2 - \left(\frac{\rho}{4} - \frac{3\rho^3}{64}\right)\\ &\leq 1 - \frac{\rho}{4} + \frac{\rho^2}{32} + \frac{3\rho^3}{64} \leq 1 - \frac{\rho}{4} + \frac{5\rho^2}{64}\end{aligned} \tag{C.30}$$

for $\rho \leq 1$. (This immediately implies the easy bound $\Upsilon \leq 1$, which follows anyhow from (5.22) for all $\rho \geq 0$.)

By (C.13),

$$\frac{j}{\upsilon^2 - \upsilon} \geq \frac{1 + \rho^2/2 - \rho^4/8}{\left(1 + \frac{\rho^2}{8}\right)^2 - \left(1 + \frac{\rho^2}{8}\right)} \geq \frac{1 + \rho^2/2 - \rho^4/8}{\frac{\rho^2}{8} + \frac{\rho^4}{64}} \geq \frac{8}{\rho^2}$$

for $\rho \leq 1$. Therefore, by (5.43),

$$\begin{aligned}\eta &\geq \frac{1}{\sqrt{2}}\sqrt{\frac{8}{\rho^2}}\left(2 - \frac{\rho}{2} + \frac{\rho^2}{8} + \frac{11\rho^3}{64} - \frac{3\rho^4}{128}\right) - \frac{1}{2}\left(1 - \frac{\rho}{4} + \frac{5\rho^2}{64}\right)^2 + \frac{1}{2} - \frac{1}{2} - \frac{\rho}{2}\\ &\geq \frac{4}{\rho} - 1 + \frac{\rho}{4} + \frac{11\rho^2}{32} - \frac{3\rho^3}{64} - \frac{1}{2}\left(1 - \frac{\rho}{2} + \frac{7\rho^2}{32}\right) - \frac{\rho}{2} \geq \frac{4}{\rho} - \frac{3}{2} + \frac{15\rho^2}{64} - \frac{3\rho^3}{64}\\ &\geq \frac{4}{\rho} - \frac{3}{2}\end{aligned}$$

for $\rho \leq 1$. This implies the bound $\eta(\rho) > 1$ for all $\rho \leq 1$. Conversely, $\eta(\rho) \geq 4/\rho - 3/2$ follows from $\eta(\rho) > 1$ for $\rho > 8/5$. We check $\eta(\rho) \geq 4/\rho - 3/2$ for $\rho \in [1, 8/5]$ by the bijection method (5 iterations).

We conclude that, for all $\rho > 0$,

$$\eta \geq \max\left(1, \frac{4}{\rho} - \frac{3}{2}\right). \tag{C.31}$$

This bound has the right asymptotics for $\rho \to 0^+$ and $\rho \to +\infty$.

Let us now bound c_0 from above. By (C.20) and (C.30),

$$\begin{aligned}c_0(\rho) = \Upsilon(\rho) \cdot \cos\theta_0 + \sin\theta_0 &\leq \left(1 - \frac{\rho}{4} + \frac{5\rho^2}{64}\right)\left(1 - \frac{\rho^2}{32} + \frac{7\rho^4}{512}\right) + \frac{\rho}{4}\\ &\leq 1 + \frac{3\rho^2}{64} + \frac{\rho^3}{128} + \frac{23\rho^4}{2048} - \frac{7\rho^5}{2048} + \frac{35\rho^6}{2^{15}} \leq 1 + \frac{\rho^2}{15}\end{aligned} \tag{C.32}$$

for $\rho \leq 1$. Since $\Upsilon \leq 1$ and $\theta_0 \in [0, \pi/4] \subset [0, \pi/2]$, the bound

$$c_0(\rho) \leq \cos\theta_0 + \sin\theta_0 \leq \sqrt{2} \tag{C.33}$$

holds for all $\rho \geq 0$. By (C.27), we also know that, for $\rho \geq 2$,

$$\begin{aligned}c_0(\rho) &\leq \left(1 - \frac{1}{\sqrt{2\rho}} + \frac{1}{\rho}\right)\cos\theta_0 + \sin\theta_0\\ &\leq \sqrt{\left(1 - \frac{1}{\sqrt{2\rho}} + \frac{1}{\rho}\right)^2 + 1} \leq \sqrt{2}\left(1 - \frac{1}{2\sqrt{2\rho}} + \frac{9}{16\rho}\right).\end{aligned} \tag{C.34}$$

From (C.31) and (C.33), we obtain that

$$\frac{1}{\eta}\left(1 + 2c_0^2\right) \leq 1 \cdot (1 + 2 \cdot 2) = 5 \tag{C.35}$$

for all $\rho \geq 0$. At the same time, (C.31) and (C.32) imply that

$$\begin{aligned}\frac{1}{\eta}\left(1 + 2c_0^2\right) &\leq \left(\frac{4}{\rho} - \frac{3}{2}\right)^{-1}\left(3 + \frac{4\rho^2}{15} + \frac{2\rho^4}{15^2}\right)\\ &= \frac{3\rho}{4}\left(1 - \frac{3\rho}{8}\right)^{-1}\left(1 + \frac{4\rho^2}{45} + \frac{\rho^4}{675}\right) \leq \frac{3\rho}{4}\left(1 + \frac{\rho}{2}\right)\end{aligned}$$

for $\rho \leq 0.4$. Hence $(1 + 2c_0^2)/\eta \leq 0.86\rho$ for $\rho < 0.29$. The bisection method (20 iterations, starting by splitting the range into 2^8 equal intervals) shows that

$(1+2c_0^2)/\eta \leq 0.86\rho$ also holds for $0.29 \leq \rho \leq 6$; for $\rho > 6$, the same inequality holds by (C.35).

We have thus shown that

$$\frac{1+2c_0^2}{\eta} \leq \min(5, 0.86\rho) \tag{C.36}$$

for all $\rho > 0$.

Now we wish to bound $\sqrt{(\upsilon^2-\upsilon)/2}$ from below. By (C.14) and (C.13),

$$\begin{aligned}\upsilon^2 - \upsilon &\geq \left(1+\frac{\rho^2}{8}-\frac{5\rho^4}{128}\right)^2 - \left(1+\frac{\rho^2}{8}\right)\\ &= 1+\frac{\rho^2}{4}-\frac{5\rho^4}{64}+\left(\frac{5\rho^2}{128}-\frac{1}{8}\right)^2\rho^4 - \left(1+\frac{\rho^2}{8}\right) \geq \frac{\rho^2}{8}-\frac{5\rho^4}{64},\end{aligned} \tag{C.37}$$

for $\rho \geq 1$, and so

$$\sqrt{\frac{\upsilon^2-\upsilon}{2}} \geq \frac{\rho}{4}\sqrt{1-\frac{5\rho^2}{8}},$$

and this is greater than $\rho/6$ for $\rho \leq 1/3$. The bisection method (20 iterations, 5 initial steps) confirms that $\sqrt{(\upsilon^2-\upsilon)/2} > \rho/6$ also holds for $2/3 < \rho \leq 4$. On the other hand, by (C.22) and $\upsilon^2 = (1+j)/2 \geq (1+\rho)/2$,

$$\begin{aligned}\sqrt{\frac{\upsilon^2-\upsilon}{2}} &\geq \sqrt{\frac{\frac{\rho+1}{2}-\sqrt{\frac{\rho}{2}}\left(1+\frac{1}{\sqrt{2}\rho}\right)}{2}} \geq \frac{\sqrt{\rho}}{2}\sqrt{1+\frac{1}{\rho}-\sqrt{\frac{2}{\rho}}\left(1+\frac{1}{\sqrt{2}\rho}\right)}\\ &\geq \frac{\sqrt{\rho}}{2}\sqrt{1-\sqrt{\frac{2}{\rho}}+\frac{1}{2\rho}} \geq \frac{\sqrt{\rho}}{2}\left(1-\sqrt{\frac{1}{2\rho}}\right) = \frac{\sqrt{\rho}}{2}-\frac{1}{2^{3/2}}\end{aligned} \tag{C.38}$$

for $\rho \geq 4$. We check by the bisection method (20 iterations) that $\sqrt{(\upsilon^2-\upsilon)/2} \geq \sqrt{\rho}/2 - 1/2^{3/2}$ also holds for all $0 \leq \rho \leq 4$.

We conclude that

$$\sqrt{\frac{\upsilon^2-\upsilon}{2}} \geq \begin{cases}\rho/6 & \text{if } \rho \leq 4,\\ \frac{\sqrt{\rho}}{2}-\frac{1}{2^{3/2}} & \text{for all } \rho.\end{cases} \tag{C.39}$$

We still have a few other inequalities to check. Let us first derive an easy lower bound on $c_1(\rho)$ for ρ large: by (C.8), (C.23) and (C.12),

$$\begin{aligned}c_1(\rho) &= \sqrt{\frac{1+1/\upsilon}{\upsilon^2-\upsilon}}\cdot\Upsilon \geq \sqrt{\frac{1}{\upsilon^2-\upsilon}}\cdot\left(1-\frac{1}{\sqrt{2}\rho}\right) \geq \sqrt{\frac{2}{\rho}+\frac{\sqrt{8}}{\rho^{3/2}}}\cdot\left(1-\frac{1}{\sqrt{2}\rho}\right)\\ &= \sqrt{\frac{2}{\rho}}\left(1+\frac{1}{\sqrt{2\rho}}-\frac{1}{4\rho}\right)\cdot\left(1-\frac{1}{\sqrt{2}\rho}\right) \geq \sqrt{\frac{2}{\rho}}\left(1-\frac{3}{4\rho}\right)\end{aligned}$$

for $\rho \geq 1$. Together with (C.34), this implies that, for $\rho \geq 2$,

$$\frac{c_0 - 1/\sqrt{2}}{\sqrt{2}c_1\rho} \leq \frac{\sqrt{2}\left(\frac{1}{2}-\frac{1}{2\sqrt{2\rho}}+\frac{9}{16\rho}\right)}{\sqrt{2}\rho\sqrt{\frac{2}{\rho}}\left(1-\frac{3}{4\rho}\right)} = \frac{1}{\sqrt{2\rho}}\cdot\frac{1-\frac{1}{\sqrt{2\rho}}+\frac{9}{8\rho}}{2\left(1-\frac{3}{4\rho}\right)},$$

again for $\rho \geq 1$. This is $\leq 1/\sqrt{8\rho}$ for $\rho \geq 8$. Hence it is $\leq 1/\sqrt{8\cdot 25} < 0.071$ for $\rho \geq 25$.

Let us now look at ρ small. By (C.13),

$$\upsilon^2 - \upsilon \leq \left(1 + \frac{\rho^2}{8}\right)^2 - \left(1 + \frac{\rho^2}{8} - \frac{5\rho^4}{32}\right) = \frac{\rho^2}{8} + \frac{9\rho^4}{32}$$

for any $\rho > 0$. Hence, by (C.15) and (C.29),

$$c_1(\rho) = \sqrt{\frac{1 + 1/\upsilon}{\upsilon^2 - \upsilon}} \cdot \Upsilon \geq \sqrt{\frac{2 - \rho^2/8}{\frac{\rho^2}{8} + \frac{9\rho^4}{32}}} \cdot \left(1 - \frac{\rho}{4}\right) \geq \frac{4}{\rho}\left(1 - \frac{5}{4}\rho^2\right)\left(1 - \frac{\rho}{4}\right),$$

whereas, for $\rho \leq 1$,

$$c_0(\rho) = \Upsilon(\rho) \cdot \cos\theta_0 + \sin\theta_0 \leq 1 + \sin\theta_0 \leq 1 + \rho/4$$

by (C.20). Thus

$$\frac{c_0 - 1/\sqrt{2}}{\sqrt{2}c_1\rho} \leq \frac{1 + \frac{\rho}{4} - \frac{1}{\sqrt{2}}}{\sqrt{2} \cdot 4\left(1 - \frac{5}{4}\rho^2\right)\left(1 - \frac{\rho}{4}\right)} \leq 0.0584$$

for $\rho \leq 0.1$. We check the remaining interval $[0.1, 25]$ (or $[0.1, 8]$, if we aim at the bound $\leq 1/\sqrt{8\rho}$) by the bisection method (with 24 iterations, including 12 initial iterations – or 15 iterations and 10 initial iterations, in the case of $[0.1, 8]$) and obtain that

$$0.0763895 \leq \max_{\rho \geq 0} \frac{c_0 - 1/\sqrt{2}}{\sqrt{2}c_1\rho} \leq 0.0763896 \tag{C.40}$$
$$\sup_{\rho \geq 0} \frac{c_0 - 1/\sqrt{2}}{c_1\sqrt{\rho}} \leq \frac{1}{2}.$$

In the same way, we see that

$$\frac{c_0}{c_1\rho} \leq \frac{1}{\sqrt{\rho}} \frac{1}{1 - \frac{3}{4\rho}} \leq 0.171$$

for $\rho \geq 36$ and

$$\frac{c_0}{c_1\rho} \leq \frac{1 + \frac{\rho}{4}}{4\left(1 - \frac{5}{4}\rho^2\right)\left(1 - \frac{\rho}{4}\right)} \leq 0.267$$

for $\rho \leq 0.1$. The bisection method applied to $[0.1, 36]$ with 24 iterations (including 12 initial iterations) now gives

$$0.29887 \leq \max_{\rho > 0} \frac{c_0}{c_1\rho} \leq 0.29888. \tag{C.41}$$

We would also like a lower bound for c_0/c_1. For c_0, we can use the lower bound $c_0 \geq 1$ given by (C.21). By (C.15), (C.30) and (C.37),

$$\begin{aligned} c_1(\rho) &= \sqrt{\frac{1 + 1/\upsilon}{\upsilon^2 - \upsilon}} \cdot \Upsilon \leq \sqrt{\frac{2 - \frac{\rho^2}{8} + \frac{7\rho^4}{128}}{\rho^2/8 - 5\rho^4/64}} \cdot \left(1 - \frac{\rho}{4} + \frac{5\rho^2}{64}\right) \\ &\leq \frac{4}{\rho}\left(1 + \frac{5\rho^2}{16}\right)\left(1 - \frac{\rho}{4} + \frac{5\rho^2}{64}\right) < \frac{4}{\rho} \end{aligned}$$

for $\rho \leq 1/4$. Thus, $c_0/(c_1\rho) \geq 1/4$ for $\rho \in [0, 1/4]$. The bisection method (with 20 iterations, including 10 initial iterations) gives us that $c_0/(c_1\rho) \geq 1/4$ also holds for $\rho \in [1/4, 6.2]$. Hence

$$\frac{c_0}{c_1} \geq \frac{\rho}{4}$$

for $\rho \leq 6.2$.

Now consider the case of large ρ. By and $\Upsilon \leq 1$,

$$\frac{c_0}{c_1\sqrt{\rho}} \geq \frac{1/\Upsilon}{\sqrt{\frac{1+1/\upsilon}{\upsilon^2-\upsilon}}\cdot\sqrt{\rho}} \geq \frac{\sqrt{(\upsilon^2-\upsilon)/\rho}}{\sqrt{1+1/\upsilon}} \geq \frac{1}{\sqrt{2}}\frac{1-1/\sqrt{2\rho}}{\sqrt{1+1/\upsilon}}. \tag{C.42}$$

(This is off from optimal by a factor of about $\sqrt{2}$.) For $\rho \geq 200$, (C.42) implies that $c_0/(c_1\sqrt{\rho}) \geq 0.6405$. The bisection method (with 20 iterations, including 5 initial iterations) gives us $c_0/(c_1\sqrt{\rho}) \geq 5/8 = 0.625$ for $\rho \in [6.2, 200]$. We conclude that

$$\frac{c_0}{c_1} \geq \min\left(\frac{\rho}{4}, \frac{5}{8}\sqrt{\rho}\right). \tag{C.43}$$

Finally, we verify an inequality that will be useful for the estimation of a crucial exponent in one of the main intermediate results (Prop. 5.1). We wish to show that, for all $\alpha \in [0, \pi/2]$,

$$\alpha - \frac{\sin 2\alpha}{4\cos^2\frac{\alpha}{2}} \geq \frac{\sin\alpha}{2\cos^2\alpha} - \frac{5\sin^3\alpha}{24\cos^6\alpha} \tag{C.44}$$

The left side is positive for all $\alpha \in (0, \pi/2]$, since $\cos^2\alpha/2 \geq 1/\sqrt{2}$ and $(\sin 2\alpha)/2$ is less than $2\alpha/2 = \alpha$. The right side is negative for $\alpha > 1$ (since it is negative for $\alpha = 1$, and $(\sin\alpha)/(\cos\alpha)^2$ is increasing on α). Hence, it is enough to check (C.44) for $\alpha \in [0, 1]$. The two sides of (C.44) are equal for $\alpha = 0$; moreover, the first four derivatives also match at $\alpha = 0$. We take the fifth derivatives of both sides; the bisection method (running on $[0, 1]$ with 20 iterations, including 10 initial iterations) gives us that the fifth derivative of the left side minus the fifth derivative of the right side is always positive on $[0, 1]$ (and minimal at 0, where it equals $30.5 + O^*\left(10^{-9}\right)$).

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